

Five-loop Konishi in $\mathcal{N} = 4$ SYM

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Abstract

We present a new method for computing the Konishi anomalous dimension in $\mathcal{N} = 4$ SYM at weak coupling. It does not rely on the conventional Feynman diagram technique and is not restricted to the planar limit. It is based on the OPE analysis of the four-point correlation function of stress-tensor multiplets, which has been recently constructed up to six loops. The Konishi operator gives the leading contribution to the singlet $SU(4)$ channel of this OPE. Its anomalous dimension is the coefficient of the leading single logarithmic singularity of the logarithm of the correlation function in the double short-distance limit, in which the operator positions coincide pairwise. We regularize the logarithm of the correlation function in this singular limit by a version of dimensional regularization. At any loop level, the resulting singularity is a simple pole whose residue is determined by a finite two-point integral with one loop less. This drastically simplifies the five-loop calculation of the Konishi anomalous dimension by reducing it to a set of known four-loop two-point integrals and two unknown integrals which we evaluate analytically. We obtain an analytic result at five loops in the planar limit and observe perfect agreement with the prediction based on integrability in AdS/CFT.

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1 Introduction

It has been realized recently that the four-point correlation function of the so-called stress-tensor multiplets in $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) has a new symmetry [1]. In combination with $\mathcal{N} = 4$ superconformal symmetry, it imposes strong constraints on the integrand of the loop correction to the correlation function and leads to an iterative structure at weak coupling, at any loop order and for a gauge group of arbitrary rank [2]. The correlation function of four stress-tensor multiplets plays a special role in $\mathcal{N} = 4$ SYM theory. In virtue of the operator product expansion (OPE), its asymptotic behaviour at short distances contains information about the anomalous dimensions of a large variety of Wilson operators and the corresponding structure constants of the OPE. Moreover, if considered in the planar limit and restricted to the light cone, it is dual to the four-particle scattering amplitudes [3, 4].

For more than ten years, this correlation function was not known beyond two loops. The main difficulty in going to higher loops is due to the factorially increasing number of contributing Feynman diagrams. In general, each individual diagram respects neither gauge invariance nor conformal symmetry, but the symmetries are restored in the sum of all diagrams. This calls for developing a new approach that makes full use of $\mathcal{N} = 4$ superconformal symmetry and of the specific symmetry of the stress-tensor multiplet mentioned above. Such an approach has been proposed in two recent papers [1, 2], where a new construction of the four-point correlation function was carried out in $\mathcal{N} = 4$ SYM theory for the gauge group $SU(N_c)$ with arbitrary N_c . In the planar limit, for $N_c \rightarrow \infty$ and with the 't Hooft coupling $a = g^2 N_c / (4\pi^2)$ fixed, the integrand of the four-point correlation function was determined up to six loops and the non-planar $O(1/N_c^2)$ correction was identified at four loops (up to four arbitrary rational constants).¹

In this paper, we apply the results of Refs. [1, 2] to perform the OPE analysis of the four-point correlation function of the stress-tensor multiplets. As the main result of our analysis, we present a new method for computing the Konishi anomalous dimension in $\mathcal{N} = 4$ SYM theory for arbitrary gauge group $SU(N_c)$. The Konishi operator is the simplest unprotected gauge invariant Wilson operator in $\mathcal{N} = 4$ SYM, whose scaling dimension receives anomalous contribution to all loops. In the OPE context, the distinguishing feature of the Konishi operator is that it controls the leading asymptotic behaviour of the four-point correlation function at loop level in the short distance limit. In this manner, we obtain an analytic result for the Konishi anomalous dimension at five loops in planar $\mathcal{N} = 4$ SYM theory and observe perfect agreement with the prediction based on integrability in AdS/CFT [5, 6, 7, 8].

The properties of the Konishi operator have been studied extensively after the discovery of the so-called Konishi anomaly [9] in supersymmetric gauge theories. The interest in the subject was renewed in the context of the AdS/CFT correspondence. As was observed in [10], the Konishi supermultiplet in $\mathcal{N} = 4$ SYM theory is a long (or unprotected) multiplet that corresponds to the first string level in the spectrum of type IIB excitations in an $AdS_5 \times S^5$ background. Recently, the Konishi anomalous dimension $\gamma_K(a)$ again attracted a lot of attention after the discovery of integrability in the planar limit on both sides of the AdS/CFT correspondence (for a review see [11]). At strong coupling, the first few terms of the expansion of $\gamma_K(a)$ in powers of $a^{-1/4}$ in the planar limit were obtained from the semiclassical quantization of short strings on an $AdS_5 \times S^5$ background [12, 13, 14]. At weak coupling, the values of $\gamma_K(a)$ at four and five loops in planar $\mathcal{N} = 4$ SYM were predicted in Refs. [5] and [6], respectively, from the integrable string

¹It is worth mentioning that our approach is not limited to six loops. Extending it to higher orders is just a question of computer power.

sigma model by evaluating finite size effects using Lüscher's formulas.² The four-loop prediction was later confirmed by direct perturbative calculations using $\mathcal{N} = 1$ Feynman super-graphs [15, 16] and ordinary Feynman diagrams [17]. Until now, no five-loop test of the integrability prediction had been performed, and it is not clear whether traditional techniques would allow one to reach such a high perturbative level. A numerical prediction for $\gamma_{\mathcal{K}}(a)$ at intermediate coupling, interpolating between the strong and weak coupling results, was made in [18] from the solution of the Y -system of integral non-linear equations and more recently in [19] from the TBA equations.

As was already mentioned, the Konishi operator provides the leading contribution to the asymptotic behaviour of the four-point correlation function at short distances $G(1, 2, 3, 4) \sim (x_{12}^2)^{\gamma_{\mathcal{K}}(a)/2}$ as $x_1 \rightarrow x_2$. At weak coupling, this asymptotic behaviour implies that perturbative corrections to the correlation function at ℓ loops are given by a sum of logarithmic singularities $(\ln x_{12}^2)^k$ with powers $k \leq \ell$. The coefficients of the higher powers of logarithms (for $k > 1$) are expressed in terms of the anomalous dimensions at lower loops. It is only the single logarithm (with $k = 1$) that carries information about the anomalous dimension at ℓ loops. This fact complicates the evaluation of the anomalous dimension. It is more advantageous to consider instead the logarithm of the correlation function $\ln G(1, 2, 3, 4)$, whose asymptotic behaviour at short distances involves a single logarithmic singularity to all loops. We can further simplify the analysis by considering the double short-distance limit $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$, in which case $\ln G(1, 2, 3, 4) \sim (\gamma_{\mathcal{K}}(a)/2) [\ln x_{12}^2 + \ln x_{34}^2]$. To determine the anomalous dimension in this way, we need an efficient way of computing the perturbative corrections to $\ln G(1, 2, 3, 4)$.

Applying the results of Ref. [2], we can express $\ln G(1, 2, 3, 4)$ at ℓ loops as a Euclidean integral, whose integrand is a conformally covariant function of the four external points and the ℓ integration points. In the short-distance limit $x_1 \rightarrow x_2$, the integral develops a single logarithmic singularity $\sim \ln x_{12}^2$ when all ℓ integration points approach the external point x_1 simultaneously. Similarly, for $x_3 \rightarrow x_4$ the logarithmic singularity $\sim \ln x_{34}^2$ originates from integration in the vicinity of the point x_3 . It is clear that these singularities are of ultraviolet (UV) origin with the small distances x_{12}^2 and x_{34}^2 playing the role of a UV cut-off. To extract the coefficient of the logarithmic singularity of the integral, which defines the anomalous dimension of the Konishi operator, we can simplify the calculation by taking $x_{12} = x_{34} = 0$ inside the integral and by introducing the most convenient regularization scheme for the resulting UV divergences. This is done by changing the integration measure from $D = 4$ to $D = 4 - 2\epsilon$ dimensions. Thus, we transform the expected single logarithmic singularity of $\ln G(1, 2, 3, 4)$ in the double short-distance limit into a simple pole $1/\epsilon$. Our final simplification comes from the observation that, for $x_1 = x_2$ and $x_3 = x_4$, the ℓ -loop residue at this simple pole is in fact given by an $(\ell - 1)$ -loop *finite* two-point integral of the propagator type. We can then apply an array of well-known and very efficient methods for computing such integrals.

In summary, we have reduced the problem of computing the Konishi anomalous dimension at ℓ loops to the problem of evaluating a *finite two-point integral at $(\ell - 1)$ loops*.³ This allowed

²See also [7, 8] for an alternative approach using the mirror thermodynamic Bethe Ansatz.

³This feature is typical for usual renormalization group calculations in momentum space [20].

us to obtain the following result up to five loops ⁴:

$$\begin{aligned} \gamma_{\mathcal{K}}(a) = & 3a - 3a^2 + \frac{21}{4}a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5 - \frac{r}{N_c^2}\zeta_5 \right) a^4 \\ & + \left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}\zeta_3^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7 + O(1/N_c^2) \right) a^5 + O(a^6), \end{aligned} \quad (1.1)$$

where the non-planar four-loop correction is predicted up to an arbitrary rational constant, r . We use ζ_n to denote values of the Riemann zeta-function $\zeta(n)$ at integer points. The obtained expression for $\gamma_{\mathcal{K}}(a)$ is in agreement with the existing perturbative four-loop results of [15, 16, 17, 26] and with the five-loop prediction of [5, 6, 7, 8] based on integrability in AdS/CFT. As a byproduct of the OPE analysis of the four-point correlation function, we also investigated the spectrum of anomalous dimensions of the twist-two operators with non-vanishing spin at three loops and found agreement with the values conjectured in [24].

The paper is organized as follows. In Section 2 we define the four-point correlation function of stress-tensor multiplets in $\mathcal{N} = 4$ SYM and use the OPE to relate $\gamma_{\mathcal{K}}(a)$ to the leading asymptotic behaviour of the correlation function in the short distance limit. In Section 3 we formulate our method for computing the Konishi anomalous dimension and illustrate it by evaluating $\gamma_{\mathcal{K}}(a)$ up to two loops. In Section 4 we extend our analysis to four loops. Applying the results of Refs. [2] for the four-loop correlation function and making use of well-known techniques for evaluating Feynman integrals, we express the four-loop correction to $\gamma_{\mathcal{K}}(a)$ as a linear combination of six master two-point three-loop integrals (five in the planar sector and only one in the non-planar sector). If rewritten in the dual momentum representation, the latter coincide with some known finite three-loop integrals of the propagator type. In Section 5 we evaluate the five-loop correction to $\gamma_{\mathcal{K}}(a)$ in the planar limit. We show that it is given by a linear combination of 22 master scalar four-loop integrals. Among them 20 integrals correspond to planar graphs and coincide, in the dual momentum representation, with known finite four-loop propagator integrals [27]. The remaining two non-planar integrals are evaluated in Appendix B. Section 6 contains concluding remarks. In Appendix A, we describe the method of IR rearrangement in the configuration space that we employ in our calculation of $\gamma_{\mathcal{K}}(a)$. In Appendix C, we perform the OPE analysis of the four-point correlation function and extract the values of three-loop anomalous dimensions of twist-two operators with Lorentz spin zero, two and four.

2 Four-point correlation function

2.1 Expression for the integrand

In this paper we study the OPE of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM. This is the simplest example of a half-BPS operator, whose superconformal primary state has the form

$$\mathcal{O}_{20'}^{IJ} = \text{tr}(\Phi^I \Phi^J) - \frac{1}{6} \delta^{IJ} \text{tr}(\Phi^K \Phi^K). \quad (2.1)$$

⁴The one-loop value of $\gamma_{\mathcal{K}}$ was found for the first time in [21]. At two loops, it was first extracted from the OPE of two stress-tensor multiplets in $\mathcal{N} = 4$ SYM, as part of the investigation of the two-loop four-point correlation functions of half-BPS operators [22, 23]. The three-loop value, together with the anomalous dimensions of all twist-two Wilson operators in $\mathcal{N} = 4$ SYM was originally conjectured in [24]. This three-loop prediction for $\gamma_{\mathcal{K}}$ was then confirmed for the first time in [25] by a direct perturbative calculation.

It is built from the six real scalars Φ^I (with $I = 1, \dots, 6$ being an $SO(6)$ index) in the adjoint representation of the gauge group $SU(N_c)$ and belongs to the representation $\mathbf{20}'$ of the R symmetry group $SO(6) \sim SU(4)$. To keep track of the $SO(6)$ tensor structure of the OPE, it proves convenient to introduce auxiliary $SO(6)$ harmonic variables Y_I , defined as a (complex) null vector, $Y^2 \equiv Y_I Y_I = 0$, and project the indices of \mathcal{O}^{IJ} as follows:

$$\mathcal{O}(x, y) \equiv Y_I Y_J \mathcal{O}_{\mathbf{20}'}^{IJ}(x) = Y_I Y_J \text{tr}(\Phi^I(x) \Phi^J(x)) , \quad (2.2)$$

where y denotes the dependence on the Y -variables.

An important property of the operator (2.1) is that its scaling dimension is protected from perturbative corrections. The same is true for the two- and three-point correlation functions of the operator $\mathcal{O}(x_i, y_i)$. The four-point correlation function is the first to receive perturbative corrections:

$$G_4 = \langle \mathcal{O}(x_1, y_1) \mathcal{O}(x_2, y_2) \mathcal{O}(x_3, y_3) \mathcal{O}(x_4, y_4) \rangle = \sum_{\ell=0}^{\infty} a^\ell G_4^{(\ell)}(1, 2, 3, 4) , \quad (2.3)$$

where the expansion on the right-hand side runs in powers of the 't Hooft coupling $a = g^2 N_c / (4\pi^2)$ and $G_4^{(\ell)}$ denotes the perturbative correction at ℓ loops. Notice that here we do not assume the planar limit and allow $G_4^{(\ell)}$ to have a non-trivial dependence on N_c . It is this four-point function that will serve as the starting point of our OPE analysis.

At tree level, $G_4^{(0)}$ reduces to a product of free scalar propagators and the corresponding expression can be found in Ref. [1]. At loop level, the superconformal symmetry of the $\mathcal{N} = 4$ SYM theory restricts $G_4^{(\ell)}$ to have the following factorized form [28, 1]:

$$G_4^{(\ell)}(1, 2, 3, 4) = \frac{2(N_c^2 - 1)}{(4\pi^2)^4} \times R(1, 2, 3, 4) \times F^{(\ell)}(x_i) , \quad (\text{for } \ell \geq 1) , \quad (2.4)$$

where $F^{(\ell)}(x_i)$ is a function of x_i only (with $i = 1, 2, 3, 4$) to be specified below and $R(1, 2, 3, 4)$ is a universal, ℓ -independent rational function of the space-time, x_i , and harmonic, Y_I , coordinates at the four external points 1, 2, 3, 4, whose explicit expression can be found in Ref. [1].

So from (2.3) and (2.4) we find that the loop corrections to the four-point correlation function are determined by a single function $F^{(\ell)}(x_i)$. As was shown in Refs. [1, 2], this function has a number of remarkable properties in $\mathcal{N} = 4$ SYM theory. Namely, it can be represented in the form of an ℓ -loop Euclidean integral,

$$F^{(\ell)}(x_i) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell! (-4\pi^2)^\ell} \int d^4 x_5 \dots d^4 x_{4+\ell} f^{(\ell)}(x_1, \dots, x_{4+\ell}) , \quad (2.5)$$

where the integrand $f^{(\ell)}$ depends on the four external coordinates x_1, \dots, x_4 and the ℓ additional (internal) coordinates $x_5, \dots, x_{4+\ell}$ giving the positions of the Lagrangian insertions. The integrand $f^{(\ell)}$ can be written in the form

$$f^{(\ell)}(x_1, \dots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \dots, x_{4+\ell})}{\prod_{1 \leq i < j \leq 4+\ell} x_{ij}^2} . \quad (2.6)$$

Here the denominator contains the product of all distances between the $(4 + \ell)$ points and $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of degree $(\ell - 1)(\ell + 4)/2$. Most importantly, this polynomial

is symmetric under the exchange of any pair of points x_i and x_j (both external and internal). As we have shown in Refs. [2], this property alone combined with the correct asymptotic behaviour of the correlation function in the short-distance and the light-cone limits, allows us to completely determine $F^{(\ell)}(x_i)$ up to six loops in the planar sector. In the non-planar sector, the same analysis leads to an expression depending on a few constants only. For example, at one and two loops, we have

$$P^{(1)} = 1, \quad P^{(2)} = \frac{1}{48} \sum_{\sigma \in S_6} x_{\sigma_1 \sigma_2}^2 x_{\sigma_3 \sigma_4}^2 x_{\sigma_5 \sigma_6}^2 = x_{12}^2 x_{34}^2 x_{56}^2 + \dots, \quad (2.7)$$

where in the second relation the sum runs over all S_6 -permutations of the indices $1, \dots, 6$. Similar expressions at higher loops can be found in Ref. [2].

2.2 Operator product expansion

As was mentioned in the previous subsection, the four-point correlation function (2.5) has a particular asymptotic behaviour at short distances, dictated by the operator product expansion (OPE).

For the scalar operators (2.2) the OPE takes the following form

$$\mathcal{O}(x_1, y_1) \mathcal{O}(x_2, y_2) = c_{\mathcal{I}} \frac{(Y_1 \cdot Y_2)^2}{x_{12}^4} \mathcal{I} + c_{\mathcal{K}}(a) \frac{(Y_1 \cdot Y_2)^2}{(x_{12}^2)^{1-\gamma_{\mathcal{K}}/2}} \mathcal{K}(x_2) + c_{\mathcal{O}} \frac{(Y_1 \cdot Y_2)}{x_{12}^2} \mathcal{O}_{20'}^{IJ}(x_2) + \dots \quad (2.8)$$

where we only displayed the contribution of operators with naive scaling dimension up to two. Here the most singular $1/x_{12}^4$ contribution comes from the identity operator \mathcal{I} while the first subleading $O(1/x_{12}^2)$ contribution originates from two operators: the half-BPS operator (2.1) and the Konishi operator defined as

$$\mathcal{K} = \text{tr} (\Phi^I \Phi^I). \quad (2.9)$$

Since the operators \mathcal{I} and $\mathcal{O}_{20'}$ are protected, the constants $c_{\mathcal{I}}$ and $c_{\mathcal{O}}$ do not depend on the coupling constant and keep their tree-level values, $c_{\mathcal{I}} = (N_c^2 - 1)/(32\pi^4)$ and $c_{\mathcal{O}} = 1/(2\pi^2)$. For the Konishi operator, both the coefficient $c_{\mathcal{K}}(a)$ and its scaling dimension $\Delta_{\mathcal{K}}(a)$ receive perturbative corrections to all loops. In what follows we will mostly concentrate on the anomalous dimension of the Konishi operator ⁵

$$\Delta_{\mathcal{K}} = 2 + \gamma_{\mathcal{K}}(a) = 2 + \sum_{\ell=1}^{\infty} a^{\ell} \gamma_{\mathcal{K}}^{(\ell)}. \quad (2.10)$$

In the singular limit $x_1 \rightarrow x_2$ we can apply the OPE expansion (2.8) to find the asymptotic behaviour of the correlation function (2.4) at short distances (in Euclidean kinematics). It receives contributions from all operators on the right-hand side of (2.8). A crucial advantage of the Konishi operator is that it has the minimal possible scaling dimension among all unprotected operators. To separate the contribution of the Konishi operator it is useful to consider the

⁵The Konishi operator is the simplest of an infinite series of twist-two operators contained in the OPE (2.8). In Appendix C we extract from the OPE the three-loop anomalous dimensions of the twist-two operators with Lorentz spin 2 and 4.

double short-distance limit $x_1 \rightarrow x_2$, $x_3 \rightarrow x_4$. Taking into account the relation (2.8) we obtain the asymptotic behaviour of the four-point correlation function in this limit as [1]

$$G_4 \xrightarrow{x_2 \rightarrow x_1, x_4 \rightarrow x_3} \frac{(N_c^2 - 1)^2}{4(4\pi^2)^4} \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2 - 1}{(4\pi^2)^4} \left[\frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4} + \frac{1}{3} \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4} (c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1) \right] + \dots, \quad (2.11)$$

where u is a conformal cross-ratio defined in (2.13) below, $y_{ij}^2 = (Y_i \cdot Y_j)$ denotes the scalar product of harmonic variables and the dots denote subleading terms.

Comparing the OPE prediction (2.11) with the general expression for the correlation function, Eqs. (2.3) and (2.4), we obtain the following relation for the functions $F^{(\ell)}(x_i)$ for $x_2 \rightarrow x_1$ and $x_4 \rightarrow x_3$

$$\sum_{\ell \geq 1} a^\ell F^{(\ell)}(x_i) \xrightarrow{x_2 \rightarrow x_1, x_4 \rightarrow x_3} \frac{1}{6x_{13}^4} (c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1) \times [1 + O(u) + O(1-v)]. \quad (2.12)$$

Here u and v are the two conformally invariant cross-ratios made of the four points x_i ,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (2.13)$$

so that $u \rightarrow 0$, $v \rightarrow 1$ in the double short-distance limit $x_2 \rightarrow x_1$, $x_4 \rightarrow x_3$. For our purposes it is convenient to introduce the notation for the function $x_{13}^4 F^{(\ell)}(x_i)$ in this limit,

$$x_{13}^4 F^{(\ell)}(x_i) \xrightarrow{x_2 \rightarrow x_1, x_4 \rightarrow x_3} \widehat{F}^{(\ell)}(x_i), \quad (2.14)$$

and to rewrite the OPE limit (2.12) as

$$\ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}^{(\ell)}(x_i) \right) \xrightarrow[u \rightarrow 0]{v \rightarrow 1} \frac{1}{2} \gamma_{\mathcal{K}}(a) \ln u + \ln(c_{\mathcal{K}}^2(a)) + O(u) + O((1-v)). \quad (2.15)$$

Let us now expand both sides of the relations (2.12) and (2.15) in the powers of the coupling a and compare their short-distance asymptotics. We find from (2.12) that $\widehat{F}^{(\ell)}(x_i) \sim (\ln u)^\ell$ as $u \rightarrow 0$. In particular, from (2.12) we have to two-loop order

$$\begin{aligned} \widehat{F}^{(1)} &= \frac{1}{12} \gamma_{\mathcal{K}}^{(1)} \ln u + \frac{1}{2} \alpha^{(1)} + \dots, \\ \widehat{F}^{(2)} &= \frac{1}{48} (\gamma_{\mathcal{K}}^{(1)})^2 (\ln u)^2 + \left(\frac{1}{12} \gamma_{\mathcal{K}}^{(2)} + \frac{1}{4} \gamma_{\mathcal{K}}^{(1)} \alpha^{(1)} \right) \ln u + \frac{1}{2} \alpha^{(2)} + \dots, \end{aligned} \quad (2.16)$$

where the constants $\gamma_{\mathcal{K}}^{(\ell)}$ and $\alpha^{(\ell)}$ define the perturbative corrections to the anomalous dimension $\gamma_{\mathcal{K}}(a) = \sum_{\ell \geq 1} a^\ell \gamma_{\mathcal{K}}^{(\ell)}$ and to the coefficients $(c_{\mathcal{K}}(a))^2 = 1 + 3 \sum_{\ell \geq 1} a^\ell \alpha^{(\ell)}$. In a similar manner, from (2.15) we obtain that the particular combination of the functions $\widehat{F}^{(\ell)}(x_i)$ with $1 \leq \ell' \leq \ell$, arising from the expansion of the logarithm on the left-hand side of (2.15), scales as $\ln u$. For instance, at two-loop order we have from (2.15)

$$\widehat{F}^{(2)} - 3(\widehat{F}^{(1)})^2 = \frac{1}{12} \gamma_{\mathcal{K}}^{(2)} \ln u + \frac{1}{2} \alpha^{(2)} - \frac{3}{4} (\alpha^{(1)})^2 + \dots \quad (2.17)$$

Comparing this relation with (2.16), we observe that the two-loop correction to the anomalous dimension $\gamma_{\mathcal{K}}^{(2)}$ appears in (2.16) in the subleading term, while in (2.16) it defines the leading singular behaviour. As we show in the next section, this property can be used to drastically simplify the calculation of the Konishi anomalous dimension $\gamma_{\mathcal{K}}(a)$.

3 Method for computing the Konishi anomalous dimension

Here we present our method for computing the Konishi anomalous dimension at higher loops. It takes full advantage of the known properties of the correlation function explained in the previous section. In this section we illustrate the key features of the method with the simplest examples of one and two loops.

Before we do this, we would like to recall the standard approach for extracting the Konishi anomalous dimension from the asymptotic logarithmic behaviour of the four-point correlation function (see, e.g., Refs. [29, 23]). With the help of the relations (2.5) – (2.7) we obtain the following expressions for the correlation function to two loops:

$$\begin{aligned} F^{(1)} &= g(1, 2, 3, 4), \\ F^{(2)} &= h(1, 2; 3, 4) + h(3, 4; 1, 2) + h(2, 3; 1, 4) + h(1, 4; 2, 3) \\ &\quad + h(1, 3; 2, 4) + h(2, 4; 1, 3) + \frac{1}{2}(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)[g(1, 2, 3, 4)]^2. \end{aligned} \quad (3.1)$$

Here the notation was introduced for the one- and two-loop conformal Euclidean integrals

$$\begin{aligned} g(1, 2, 3, 4) &= -\frac{1}{4\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}, \\ h(1, 2; 3, 4) &= \frac{x_{34}^2}{(4\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)}, \end{aligned} \quad (3.2)$$

with the remaining h -integrals obtained by permuting the indices.

The explicit expressions for these integrals as functions of the conformal ratios (2.13) are known [30], but what we need here is just their asymptotic behaviour for $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$, or equivalently $u \rightarrow 0$ and $v \rightarrow 1$. Replacing the integrals in (3.1) by their asymptotic expansions, we easily obtain the following result for the one- and two-loop correlation functions in the singular short-distance limit:

$$\begin{aligned} \widehat{F}^{(1)} &= \frac{1}{4} \ln u - \frac{1}{2} + \dots, \\ \widehat{F}^{(2)} &= \frac{3}{16} (\ln u)^2 - \ln u + \frac{3}{4} \zeta_3 + \frac{7}{4} + \dots, \end{aligned} \quad (3.3)$$

leading to

$$\widehat{F}^{(2)} - 3(\widehat{F}^{(1)})^2 = -\frac{1}{4} \ln u + \frac{3}{4} \zeta_3 + 1 + \dots \quad (3.4)$$

These relations are in perfect agreement with the OPE prediction (2.16) and (2.17). They allow us to reproduce the well-known result for the two-loop Konishi anomalous dimension, Eq. (1.1), and the two-loop normalization coefficients, $\alpha^{(1)} = -1$ and $\alpha^{(2)} = 3\zeta_3/2 + 7/2$.

3.1 One loop

Let us now return to the one-loop expression $\widehat{F}^{(1)}$, Eq. (3.3), and understand the origin of the singularity $\widehat{F}^{(1)} \sim \ln u$ at short distances. It is easy to see from (3.2) that for $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$ the integral $g(1, 2, 3, 4)$ develops a logarithmic divergence coming from the two distinct integration regions $x_5 \rightarrow x_1$ and $x_5 \rightarrow x_3$,

$$\widehat{F}^{(1)} \sim -\frac{1}{4\pi^2} \int_{x_{51}^2 < \delta^2} \frac{d^4 x_5}{x_{15}^2 x_{25}^2} - \frac{1}{4\pi^2} \int_{x_{53}^2 < \delta^2} \frac{d^4 x_5}{x_{35}^2 x_{45}^2}. \quad (3.5)$$

Here we have restricted the integration to two balls of radius δ , centred at the points $x_1 \sim x_2$ and $x_3 \sim x_4$. Choosing $x_{13}^2 \gg \delta^2 \gg x_{12}^2, x_{34}^2$ allows us to replace the other two propagator factors in the first integral $x_{35}^2 \sim x_{45}^2$ by x_{13}^2 , and similarly for the second. This simplification of the integrand by replacing it with its asymptotic expression in the relevant integration region will be very helpful at higher loops. Going to radial coordinates we find

$$\widehat{F}^{(1)} \sim -\frac{1}{4} \int_{x_{12}^2}^{\delta^2} \frac{dx_{51}^2}{x_{51}^2} - \frac{1}{4} \int_{x_{34}^2}^{\delta^2} \frac{dx_{53}^2}{x_{53}^2} = \frac{1}{4} \ln(x_{12}^2 x_{34}^2 / \delta^4) + \dots, \quad (3.6)$$

where the short distances $x_{12}^2, x_{34}^2 \rightarrow 0$ serve as UV cut-offs and the dots denote terms finite in the limit $x_{12}^2, x_{34}^2 \rightarrow 0$. It is easy to see that this relation is in agreement with the first relation in (3.3).

Let us now examine what happens if we interchange the integration in (3.5) with taking the limit $x_1 \rightarrow x_2, x_3 \rightarrow x_4$. In this limit, the first integral on the right-hand side of (3.5) reduces to $\int_{x_{51}^2 < \delta^2} d^4 x_5 / x_{51}^4$ and it diverges as $x_5 \rightarrow x_1$. This is not surprising since x_{12}^2 plays the role of a short-distance cut-off in the first integral in (3.5) and (3.6). Therefore, in order to define the integral for $x_{12}^2 = 0$ we have to introduce a different short-distance regulator. The simplest way to do this is to modify the integration measure in (3.5) as follows ⁶

$$d^4 x_5 \rightarrow \mu^{2\epsilon} d^{4-2\epsilon} x_5, \quad (\text{with } \epsilon < 0), \quad (3.7)$$

without changing the form of the integrand. In this way, we find from (3.5)

$$\widehat{F}_\epsilon^{(1)} \sim -\frac{\mu^{2\epsilon}}{4\pi^2} \int_{x_{51}^2 < \delta^2} \frac{d^{4-2\epsilon} x_5}{(x_{51}^2)^2} - \frac{\mu^{2\epsilon}}{4\pi^2} \int_{x_{53}^2 < \delta^2} \frac{d^{4-2\epsilon} x_5}{(x_{53}^2)^2}, \quad (3.8)$$

where we introduced the subscript in $\widehat{F}_\epsilon^{(1)}$ to indicate that it is defined in the regularization scheme (3.7) at $x_{12}^2 = x_{34}^2 = 0$. Going to spherical coordinates, $d^{4-2\epsilon} x = S_\epsilon r^{3-2\epsilon} dr$ with $S_\epsilon = 2\pi^{2-\epsilon}/\Gamma(2-\epsilon)$, we find that the two integrals in (3.8) produce equal contributions, leading to

$$\widehat{F}_\epsilon^{(1)} = \frac{(\delta^2/\mu^2)^{-\epsilon}}{2\epsilon} + O(\epsilon^0) = \frac{1}{2\epsilon} + \frac{1}{2} \ln(\mu^2/\delta^2) + \dots. \quad (3.9)$$

Comparing the right-hand sides of the relations (3.6) and (3.9), we observe that they coincide (up to the $O(1/\epsilon)$ term) upon the identification $x_{12}^2 \rightarrow \mu^2$ and $x_{34}^2 \rightarrow \mu^2$. In other words, for

⁶We would like to emphasize that this regularization is different from the conventional dimensional regularization in coordinate space in the sense that we modify the integration measure only and use the scalar propagators $1/x^2$ instead of $1/(x^2)^{1-\epsilon}$. This explains why ϵ should be kept negative.

$x_{12}^2 = x_{34}^2 = 0$, within the regularization scheme (3.7), the dimensionful parameter μ^2 plays the role of the UV cut-off. This property allows us to relate the coefficient in front of $\ln u$ in the asymptotic behaviour of $F^{(1)}$ at small u , Eq. (3.3), to the residue of $F_\epsilon^{(1)}$ at the simple pole $1/\epsilon$. Moreover, it follows from (2.16) that this coefficient coincides with the one-loop Konishi anomalous dimension $\gamma_{\mathcal{K}}^{(1)}$, leading to

$$\gamma_{\mathcal{K}}^{(1)} = 12 \frac{d}{d \ln u} \widehat{F}^{(1)} = 6 \frac{d}{d \ln \mu^2} \widehat{F}_\epsilon^{(1)} = 3, \quad (3.10)$$

in agreement with (1.1).

This suggests a new method for computing the Konishi anomalous dimension: Instead of evaluating the finite four-dimensional integrals in (3.1) and finding their asymptotics at $u \rightarrow 0, v \rightarrow 1$ afterwards, we can first evaluate the integrand at $x_1 = x_2$ and $x_3 = x_4$, thus making the integrals divergent, then introduce the regularization (3.7) and, finally, identify the terms singular for $\epsilon \rightarrow 0$.

We would like to emphasize that this method captures correctly only the terms divergent for $u \rightarrow 0, v \rightarrow 1$ but not the finite ones. To see this, let us apply the above procedure to the one-loop expression $\widehat{F}^{(1)}$

$$\widehat{F}^{(1)} \xrightarrow{x_{12}, x_{34}=0} \widehat{F}_\epsilon^{(1)} = -\frac{\mu^{2\epsilon}}{4\pi^2} \int \frac{d^{4-2\epsilon} x_5 x_{13}^4}{x_{15}^4 x_{35}^4}. \quad (3.11)$$

In comparison with (3.8) here we did not restrict the integration region over x_5 . This is not really necessary, since the integral converges at large x_5 . To perform the integration, it is convenient to switch to the dual momenta $k = x_{15}$ and $p = x_{13}$. Then, the integral in (3.11) takes the form of the standard one-loop “bubble” momentum integral of propagator type:

$$M^{(1)} = -\frac{\mu^{2\epsilon}}{4\pi^2} \int \frac{d^{4-2\epsilon} k p^4}{k^4 (p-k)^4}, \quad (3.12)$$

leading to (with $\bar{\mu}^2 = \mu^2 / (e^{\gamma_E} \pi)$)⁷

$$\widehat{F}_\epsilon^{(1)} = M^{(1)} = (x_{13}^2 / \bar{\mu}^2)^{-\epsilon} \left(\frac{1}{2\epsilon} + \frac{1}{2} + O(\epsilon^2) \right). \quad (3.13)$$

Here the pole in ϵ comes from the integration over small momenta $k \rightarrow 0$ and $(p-k) \rightarrow 0$ and, therefore, has an IR origin in the dual momentum representation. Comparison of (3.13) with the first relation in (3.3) shows that the singular term is reproduced correctly (upon identifying $x_{12}^2, x_{34}^2 \rightarrow \mu^2$ and subtracting the pole), while the regular (constant) term is different.

3.2 Two loops

Let us now extend the above analysis to two loops. According to (3.3), the two-loop correction to the correlation function $\widehat{F}^{(2)}$ has a stronger, $(\ln u)^2$ singularity for $u \rightarrow 0$. The reason for this is that, in the short distance limit $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$, the integrals on the right-hand side of (3.1) develop overlapping singularities when the integration points x_5 and x_6 independently approach the two external points, e.g. $x_5 \rightarrow x_1$ and $x_6 \rightarrow x_3$. At the same time, the leading $(\ln u)^2$

⁷In what follows, for the sake of simplicity we do not distinguish between $\bar{\mu}^2$ and μ^2 .

singularity is supposed to cancel in the particular combination of one- and two-loop corrections (3.4), which defines the $O(a^2)$ correction to the logarithm of the correlation function on the left-hand side of (2.15).

To understand the reason for this, we replace $\widehat{F}_\epsilon^{(1)}$ and $\widehat{F}_\epsilon^{(2)}$ on the left-hand side of (3.4) by their explicit expressions (3.1) and, then, simplify the resulting expression by applying the same limiting procedure as in (3.11). Namely, we take the limit $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$ inside the g - and h -integrals and modify the integration measure as in (3.7). In this manner, we arrive at

$$\widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2 = \left(\frac{\mu^{2\epsilon}}{4\pi^2}\right)^2 \int d^{4-2\epsilon}x_5 d^{4-2\epsilon}x_6 \frac{2x_{13}^6(x_{15}^2x_{36}^2 + x_{16}^2x_{35}^2 - x_{13}^2x_{56}^2)}{(x_{15}^4x_{16}^4)x_{56}^2(x_{35}^4x_{36}^4)}, \quad (3.14)$$

where the expression on the right-hand side is manifestly symmetric with respect to the integration points, x_5 and x_6 , and it takes into account the contribution from the sum of g^2 - and h -integrals. It is clear from (3.14) that the integral diverges logarithmically when x_5 and x_6 approach the external points x_1 and x_3 . The simplest way to evaluate (3.14) is by going to the dual momenta $k_1 = x_{15}$, $k_2 = x_{16}$ and $p = x_{13}$, so that

$$\widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2 = 4M^{(2)} - 2(M^{(1)})^2. \quad (3.15)$$

Here the one-loop integral $M^{(1)}$ was introduced in (3.12) and $M^{(2)}$ stands for the standard two-loop scalar propagator-type integral [31]

$$\begin{aligned} M^{(2)} &= \left(\frac{\mu^{2\epsilon}}{4\pi^2}\right)^2 \int \frac{d^{4-2\epsilon}k_1 d^{4-2\epsilon}k_2}{k_1^4 k_2^2 (k_1 - k_2)^2 (p - k_1)^2 (p - k_2)^4} \\ &= (x_{13}^2/\mu^2)^{-2\epsilon} \left(\frac{1}{8\epsilon^2} + \frac{3}{16\epsilon} - \frac{1}{16} + O(\epsilon) \right). \end{aligned} \quad (3.16)$$

Substituting (3.13) and (3.16) into (3.15), we find that the double pole cancels leading to

$$\widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2 = (x_{13}^2/\mu^2)^{-2\epsilon} \left(-\frac{1}{4\epsilon} - \frac{3}{4} + O(\epsilon) \right). \quad (3.17)$$

We observe that the expansion of this expression around $\epsilon = 0$ produces the logarithmic term $-(1/2)\ln(\mu^2/x_{13}^2)$, which matches the $-(1/4)\ln u$ term on the right-hand side of (3.4) after the identification $x_{12}^2, x_{34}^2 \rightarrow \mu^2$. As in the one-loop case, the finite terms in the two expressions are different. According to (2.17), the coefficient in front of $\ln u$ is related to the two-loop Konishi anomalous dimension, $\gamma_K^{(2)}/12$. Similarly to (3.10), this allows us to write

$$\gamma_K^{(2)} = 6 \frac{d}{d \ln \mu^2} [\widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2] = -3, \quad (3.18)$$

in agreement with (1.1).

What is the reason why the double pole cancels in (3.14)? This happens because the numerator in (3.14) has the following characteristic feature: it vanishes for $x_5 \rightarrow x_1$ and $x_5 \rightarrow x_3$ with x_6 in general position. As a consequence, the most singular contribution coming from the two regions, $x_5 \rightarrow x_1, x_6 \rightarrow x_3$ and $x_5 \rightarrow x_3, x_6 \rightarrow x_1$, is suppressed and the integral in (3.14) develops a weaker singularity. It only arises when the two integration points approach one of the

external points simultaneously, $x_5, x_6 \rightarrow x_1$ and $x_5, x_6 \rightarrow x_3$. We can make use of this fact to further simplify the calculation of the divergent part of the integral (3.14).

Like in the one-loop case (3.5), we can single out the divergent contribution to (3.14) by introducing a dimensionful parameter $\delta^2 \ll x_{13}^2$ and restricting the integration in (3.14) to a ball of radius δ centred at the points x_1 or x_3 . Due to the symmetry of the integral (3.14) under the exchange of x_1 and x_3 , the two regions produce the same contribution leading to

$$\begin{aligned} \widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2 &\sim 4 \left(\frac{\mu^{2\epsilon}}{4\pi^2} \right)^2 \int_{\Omega_\delta} d^{4-2\epsilon} x_5 d^{4-2\epsilon} x_6 \frac{x_{15}^2 + x_{16}^2 - x_{56}^2}{(x_{15}^4 x_{16}^4) x_{56}^2} \\ &= \frac{1}{2} \left(\frac{\mu^{2\epsilon}}{\pi^2} \right)^2 \int_{\Omega_\delta} d^{4-2\epsilon} x_5 d^{4-2\epsilon} x_6 \frac{(x_{15} \cdot x_{16})}{(x_{15}^4 x_{16}^4) x_{56}^2}. \end{aligned} \quad (3.19)$$

Here the integration is performed over the region Ω_δ defined as $x_{51}^2, x_{61}^2 < \delta^2$. Notice that in this region we can safely replace x_{35}^2 and x_{36}^2 inside the integral with x_{13}^2 . To simplify (3.19), we introduce the radial coordinates $r_5^2 = x_{51}^2$, $r_6^2 = x_{61}^2$ and the angle ϕ between the two vectors, $(x_{15} \cdot x_{16}) = r_5 r_6 \cos \phi$. Integration over the angle yields [32]

$$\text{r.h.s. of (3.19)} = \mu^{4\epsilon} \int_0^\delta \frac{dr_5 dr_6}{(r_5 r_6)^{2\epsilon}} \frac{r_{<}}{r_{>}^3} = (\delta^2/\mu^2)^{-2\epsilon} \left(-\frac{1}{4\epsilon} + O(\epsilon^0) \right), \quad (3.20)$$

where the notation was introduced for $r_{<} = \min(r_5, r_6)$ and $r_{>} = \max(r_5, r_6)$. As expected, the residue of the pole in (3.20) is the same as in (3.17) and, therefore, it leads to the same result for the two-loop Konishi anomalous dimension (3.18).

3.3 Loop reduction

Comparing the integrals in Eqs. (3.14) and (3.19), we notice that the latter contains a smaller number of propagators and, therefore, is much easier to analyze. Still, both integrals are two-loop ones as they involve integration over two points. As we show in this subsection, there is yet another simplification which allows us to effectively eliminate the integration over one point and, therefore, reduce the number of loops by one.

Let us return to the relation (3.19) and perform the integration over x_6 with the point x_5 in a general position inside the region Ω_δ . It is easy to see that the x_6 -integral depends on the two scales x_{15}^2 and δ^2 , which play the role of cut-offs at short and large distances, respectively. Notice that the integral converges at large x_6 and, therefore, it would stay finite if we sent δ^2/x_{15}^2 to infinity. This means that, as far as the leading divergence of (3.19) is concerned, when computing the x_6 -integral we can neglect its δ^2 -dependence and extend the integration over x_6 to the whole $(4-2\epsilon)$ -dimensional (Euclidean) space

$$\begin{aligned} \frac{1}{\pi^2} \int_{\Omega_\delta} d^{4-2\epsilon} x_6 \frac{(x_{15} \cdot x_{16})}{x_{16}^4 x_{56}^2} &= \frac{1}{\pi^2} \int d^{4-2\epsilon} x_6 \frac{(x_{15} \cdot x_{16})}{x_{16}^4 x_{56}^2} \times [1 + O(x_{15}^2/\delta^2)] \\ &= C_\epsilon (x_{15}^2)^{-\epsilon} \times [1 + O(x_{15}^2/\delta^2)]. \end{aligned} \quad (3.21)$$

Here the second relation follows from dimensional analysis of the integral and C_ϵ is some constant regular at $\epsilon = 0$. Substituting this result on the right-hand side of (3.19) we arrive at the integral

$$\frac{C_\epsilon \mu^{4\epsilon}}{2\pi^2} \int_{\Omega_\delta} \frac{d^{4-2\epsilon} x_5}{(x_{15}^2)^{2+\epsilon}} [1 + O(x_{15}^2/\delta^2)]. \quad (3.22)$$

$$\begin{array}{c} \alpha \qquad \qquad \beta \\ \hline \bullet \end{array} = \frac{\alpha+\beta-D/2}{\hline} \times G(\alpha, \beta)$$

Figure 1: Diagrammatic representation of the chain relation. Solid line with index α stands for $1/(x^2)^\alpha$ and black dot denotes the integration point.

It is easy to see that the $O(x_{15}^2/\delta^2)$ term does not produce a pole in ϵ and, therefore, can be safely neglected. Performing the x_5 -integration, we finally obtain

$$\widehat{F}_\epsilon^{(2)} - 3(\widehat{F}_\epsilon^{(1)})^2 \sim (\delta^2/\mu^2)^{-2\epsilon} \left(-\frac{C_\epsilon}{4\epsilon} + O(\epsilon^0) \right). \quad (3.23)$$

Thus, the residue at the pole and, hence, the *two-loop* Konishi anomalous dimension, is determined by the constant C_ϵ which is given in its turn by the *one-loop* integral (3.21)

$$C_\epsilon = \frac{(x_{15}^2)^\epsilon}{\pi^2} \int d^{4-2\epsilon} x_6 \frac{(x_{15} \cdot x_{16})}{x_{16}^4 x_{56}^2} = 1 + O(\epsilon). \quad (3.24)$$

Substituting this relation into (3.23), we arrive at (3.20).

The key point in the above argument is that the final result in (3.24) is given by an integral which is *finite* for $\epsilon \rightarrow 0$. Indeed, the potential singularities of this integral could come from integration in the vicinity of x_1 (at infinity it is obviously finite by power counting). However, the numerator of the integrand vanishes for $x_6 \rightarrow x_1$, so the integral is convergent. Notice that the relation (3.24) can be rewritten in a form resembling the first line in (3.19):

$$C_\epsilon = \frac{(x_{15}^2)^\epsilon}{2\pi^2} \left[\int \frac{d^{4-2\epsilon} x_6 x_{15}^2}{x_{16}^4 x_{56}^2} + \int \frac{d^{4-2\epsilon} x_6}{x_{16}^2 x_{56}^2} - \int \frac{d^{4-2\epsilon} x_6}{x_{16}^4} \right]. \quad (3.25)$$

In this form, each integral in the square brackets develops a pole $1/\epsilon$ but their sum is finite for $\epsilon \rightarrow 0$. The first integral diverges when $x_6 \rightarrow x_1$, the second one when $x_6 \rightarrow \infty$. Both of them can be easily evaluated with the help of the “chain relation” [32] shown diagrammatically in Fig. 1:

$$\begin{aligned} \frac{1}{\pi^{D/2}} \int \frac{d^D x_0}{(x_{10}^2)^\alpha (x_{02}^2)^\beta} &= \frac{G(\alpha, \beta)}{(x_{12}^2)^{\alpha+\beta-D/2}}, \\ G(\alpha, \beta) &= \frac{\Gamma(\alpha + \beta - D/2) \Gamma(D/2 - \alpha) \Gamma(D/2 - \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(D - \alpha - \beta)}. \end{aligned} \quad (3.26)$$

It is easy to check that the pole $1/\epsilon$ cancels in the sum of two integrals. The third integral in (3.25) is scaleless and, therefore, it vanishes in dimensional regularization.⁸

3.4 The method

We are now ready to formulate our method for computing the Konishi anomalous dimension at higher loops. It consists of four steps:

⁸More precisely, it develops two poles $1/\epsilon$ after integration over $x_6 \rightarrow x_1$ and $x_6 \rightarrow \infty$, one of UV and the other of IR origin. Their residues have opposite signs, so they cancel in the sum.

Step 1: Expand the logarithm of the correlation function (2.15) in powers of the coupling constant and obtain the ℓ -loop correction to the left-hand side of (2.15) in the form of an ℓ -folded integral over the internal points $x_5, \dots, x_{4+\ell}$:

$$\ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}^{(\ell)}(x_i) \right) = \sum_{\ell \geq 1} a^\ell \int d^4 x_5 \dots d^4 x_{4+\ell} \mathcal{I}_\ell(x_1, \dots, x_4 | x_5, \dots, x_{4+\ell}), \quad (3.27)$$

with the integrand \mathcal{I}_ℓ symmetric under the $S_4 \times S_\ell$ permutations of the four external points, x_1, \dots, x_4 and the ℓ integration points $x_5, \dots, x_{4+\ell}$. In the short distance limit, for $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$, the integral develops a single logarithmic singularity.

Step 2: Replace the integrand \mathcal{I}_ℓ by its limiting value at $x_1 = x_2$ and $x_3 = x_4$, introduce the regularization (3.7) for the integration measure over the ℓ internal points and, then, restrict the integration over $x_5, \dots, x_{4+\ell}$ to a ball of radius δ centred at one of the external points, say, x_1 :

$$2 \times \sum_{\ell \geq 1} a^\ell (\mu^2)^{\ell\epsilon} \int_{\Omega_\delta} d^{4-2\epsilon} x_5 \dots d^{4-2\epsilon} x_{4+\ell} \widehat{\mathcal{I}}_\ell(x_1 | x_5 \dots, x_{4+\ell}). \quad (3.28)$$

Here we inserted the factor of 2 to take into account the contribution from the integration around the point x_3 , and introduced the notation for

$$\lim_{\substack{x_2 \rightarrow x_1 \\ x_4 \rightarrow x_3}} \mathcal{I}_\ell(x_1, \dots, x_4 | x_5 \dots, x_{4+\ell}) = \widehat{\mathcal{I}}_\ell(x_1 | x_5 \dots, x_{4+\ell}) + O(\delta^2/x_{13}^2). \quad (3.29)$$

Step 3: Freeze one of the integration points, say, x_5 and perform the integration over the remaining $(\ell - 1)$ points by extending the integration region to the whole space:

$$2 \int d^{4-2\epsilon} x_6 \dots d^{4-2\epsilon} x_{4+\ell} \widehat{\mathcal{I}}_\ell(x_1 | x_5 \dots, x_{4+\ell}) = \frac{C_{\ell-1}}{\pi^2} (x_{15}^2)^{-2-(\ell-1)\epsilon}, \quad (3.30)$$

with the constant $C_{\ell-1}$ being regular at $\epsilon = 0$. Going back to (3.27), we perform the remaining x_5 -integration and obtain:

$$\begin{aligned} \ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}_\epsilon^{(\ell)}(x_i) \right) &\sim \sum_{\ell \geq 1} a^\ell C_{\ell-1} \frac{(\mu^2)^{\ell\epsilon}}{\pi^2} \int_{\Omega_\delta} \frac{d^{4-2\epsilon} x_5}{(x_{15}^2)^{2+(\ell-1)\epsilon}} \\ &= - \sum_{\ell \geq 1} a^\ell C_{\ell-1} \frac{1}{\ell\epsilon} (\mu^2/\delta^2)^{\ell\epsilon} + O(\epsilon^0). \end{aligned} \quad (3.31)$$

The expansion of this relation in the powers of ϵ produces a $\ln \mu^2$ term which should match the $\ln u$ term on the right-hand side of (2.15) for $x_{12}^2, x_{34}^2 \rightarrow \mu^2$.

Step 4: We compare the last relation with (2.15), identify $\ln u \rightarrow 2 \ln(\mu^2/\delta^2)$ and obtain the Konishi anomalous dimension as

$$\gamma_\kappa(a) = \frac{d}{d \ln \mu^2} \ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}_\epsilon^{(\ell)}(x_i) \right) \quad (3.32)$$

leading to

$$\gamma_{\mathcal{K}}(a) = - \sum_{\ell \geq 1} a^\ell C_{\ell-1}. \quad (3.33)$$

This relation allows us to express the Konishi anomalous dimension at ℓ loops in terms of the constants $C_{\ell-1}$ which are defined in their turn in terms of scalar integrals at $(\ell - 1)$ loops, Eq. (3.30).

As we will show in this paper, going through these steps we will be able to determine the Konishi anomalous dimension up to five loops in the planar sector, as well as at four loops in the non-planar sector (up to a rational prefactor, see below).

4 Konishi anomalous dimension at three and four loops

Let us apply the method described in the previous section to compute the Konishi anomalous dimension at three loops and beyond.

4.1 Preliminaries

We start with the general expression for the logarithm of the correlation function in the short-distance limit on the left-hand side of (3.27),

$$\ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}^{(\ell)} \right) = \sum_{\ell \geq 1} a^\ell I^{(\ell)}, \quad (4.1)$$

where $I^{(\ell)}$ are given by the following expressions up to five loops:

$$\begin{aligned} I^{(1)} &= 6 \widehat{F}^{(1)}, \\ I^{(2)} &= 6 [\widehat{F}^{(2)} - 3(\widehat{F}^{(1)})^2], \\ I^{(3)} &= 6 [\widehat{F}^{(3)} - 6\widehat{F}^{(1)}\widehat{F}^{(2)} + 12(\widehat{F}^{(1)})^3], \\ I^{(4)} &= 6 [\widehat{F}^{(4)} - 6\widehat{F}^{(1)}\widehat{F}^{(3)} - 3(\widehat{F}^{(2)})^2 + 36\widehat{F}^{(2)}(\widehat{F}^{(1)})^2 - 54(\widehat{F}^{(1)})^4], \\ I^{(5)} &= 6 [\widehat{F}^{(5)} - 6\widehat{F}^{(1)}\widehat{F}^{(4)} - 6\widehat{F}^{(3)}\widehat{F}^{(2)} + 36\widehat{F}^{(3)}(\widehat{F}^{(1)})^2 \\ &\quad + 36\widehat{F}^{(1)}(\widehat{F}^{(2)})^2 - 216\widehat{F}^{(2)}(\widehat{F}^{(1)})^3 + \frac{1296}{5}(\widehat{F}^{(1)})^5]. \end{aligned} \quad (4.2)$$

We recall that the function $\widehat{F}^{(\ell)}$ was defined in (2.14) as the short-distance limit of the correlation function at ℓ loops, Eq. (2.5).

To find $\widehat{F}^{(\ell)}$, we substitute (2.5) into (2.14) and regularize the integration measure according to (3.7):

$$\widehat{F}_\epsilon^{(\ell)}(x_i) = \frac{x_{13}^4 \mu^{2\ell\epsilon}}{\ell! (-4\pi^2)^\ell} \int \frac{d^{4-2\epsilon}x_5 \dots d^{4-2\epsilon}x_{4+\ell}}{(x_{15}^4 x_{35}^4) \dots (x_{1,4+\ell}^4 x_{3,4+\ell}^4)} \frac{\widehat{P}^{(\ell)}(x_1, x_3; x_5, \dots, x_{4+\ell})}{\prod_{5 \leq i < j \leq 4+\ell} x_{ij}^2}. \quad (4.3)$$

Here $\widehat{P}^{(\ell)}$ coincides with the polynomial $P^{(\ell)}$, Eq. (2.6), evaluated at $x_2 = x_1$ and $x_4 = x_3$

$$\widehat{P}^{(\ell)} = \lim_{\substack{x_2 \rightarrow x_1 \\ x_4 \rightarrow x_3}} P^{(\ell)}(x_1, \dots, x_{4+\ell}). \quad (4.4)$$

For instance, for $\ell = 1$ and $\ell = 2$ we find from (2.7)

$$\widehat{P}^{(1)} = 1, \quad \widehat{P}^{(2)} = 2x_{13}^4 x_{56}^2 + 4x_{13}^2 x_{15}^2 x_{36}^2 + 4x_{13}^2 x_{16}^2 x_{35}^2. \quad (4.5)$$

Notice that the polynomial $P^{(\ell)}$ is symmetric with respect to all $(4 + \ell)$ points while for the polynomial $\widehat{P}^{(\ell)}$ this symmetry reduces to $S_2 \times S_\ell$ permutations of two external points x_1, x_3 and of the ℓ integration points $x_5, \dots, x_{4+\ell}$.

Substituting the definition (4.3) into (4.2), we can represent the right-hand side of (4.1) in the same form as in (3.27), in terms of the ℓ -fold integrals

$$I^{(\ell)} = \mu^{2\ell\epsilon} \int d^{4-2\epsilon} x_5 \dots d^{4-2\epsilon} x_{4+\ell} \mathcal{I}_\ell, \quad (4.6)$$

and express the integrands \mathcal{I}_ℓ in terms of the polynomials $\widehat{P}^{(\ell')}$ with $1 \leq \ell' \leq \ell$. For $\ell = 1$ and $\ell = 2$ we have

$$\begin{aligned} \mathcal{I}_1 &= -\frac{3}{2\pi^2} \frac{x_{13}^4}{x_{15}^4 x_{35}^4} \\ \mathcal{I}_2 &= \frac{3}{(4\pi^2)^2} \frac{x_{13}^4}{x_{15}^4 x_{35}^4 x_{16}^4 x_{36}^4} \left(\frac{\widehat{P}_{5,6}}{x_{56}^2} - 6x_{13}^4 \right) = \frac{3}{4\pi^4} \frac{x_{13}^6 (x_{15}^2 x_{36}^2 + x_{16}^2 x_{35}^2 - x_{13}^2 x_{56}^2)}{x_{15}^4 x_{35}^4 x_{16}^4 x_{36}^4 x_{56}^2}, \end{aligned} \quad (4.7)$$

where $\widehat{P}_{5,6} \equiv \widehat{P}^{(2)}(x_5, x_6)$. It is easy to see that the numerator of \mathcal{I}_2 vanishes for $x_5 \rightarrow x_1$ and $x_5 \rightarrow x_3$, as needed to achieve a lower degree of divergence.

As was already explained, the integral in (4.6) develops a *simple* pole $O(1/\epsilon)$ from integration over all points $x_5, \dots, x_{4+\ell}$ in the vicinity of the two external points x_1 and x_3 . For $x_i \rightarrow x_1$ we can safely replace $x_{3i}^2 \rightarrow x_{13}^2$ inside \mathcal{I}_ℓ without affecting the residue at the pole. In this way, we construct the function $\widehat{\mathcal{I}}_\ell$ defined in (3.29). At one and two loops we have

$$\widehat{\mathcal{I}}_1 = -\frac{3}{2\pi^2} \frac{1}{x_{15}^4}, \quad \widehat{\mathcal{I}}_2 = \frac{3}{4\pi^4} \frac{(x_{15}^2 + x_{16}^2 - x_{56}^2)}{x_{15}^4 x_{16}^4 x_{56}^2}. \quad (4.8)$$

By construction, the functions $\widehat{\mathcal{I}}_\ell$ do not depend on x_3 . Substituting these relations in (3.30), we find with the help of (3.24)

$$C_0 = -3, \quad C_1 = 3C_\epsilon = 3, \quad (4.9)$$

with C_ϵ defined in (3.24).

4.2 Three loops

According to (2.5) and (2.6) the correlation function at three loops $F^{(3)}(x_i)$ is determined by the S_7 -invariant polynomial $P^{(3)}(x_1, \dots, x_7)$. As was shown in Ref. [2], the form of this polynomial can be fixed from the requirement for the correlation function to have the correct asymptotic behaviour at short distances. The result is

$$P^{(3)} = \frac{1}{20} (x_{12}^2)^2 (x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) + S_7 \text{ permutations}, \quad (4.10)$$

where the sum runs over the S_7 permutations of the indices $1, \dots, 7$. The explicit expression for $P^{(3)}$ contains 5040 distinct terms. However, as was explained above, for our purposes we only need its limit (4.4) for $x_2 = x_1$ and $x_4 = x_3$. This brings the number of terms down to 27:

$$\begin{aligned} \widehat{P}^{(3)} = & 8x_{13}^2x_{16}^2x_{17}^4x_{35}^4x_{36}^2 + 4x_{13}^4x_{16}^2x_{17}^2x_{35}^2x_{37}^2x_{56}^2 + 4x_{13}^6x_{17}^2x_{36}^2x_{56}^2x_{57}^2 \\ & + 2x_{13}^4x_{17}^4x_{35}^2x_{36}^2x_{56}^2 + 2x_{13}^4x_{15}^2x_{16}^2x_{37}^2x_{56}^2 + 2x_{13}^6x_{17}^2x_{37}^2x_{56}^4 + S_3 \text{ permutations}, \end{aligned} \quad (4.11)$$

where the S_3 permutations only act on the integration points x_5, x_6, x_7 . The polynomial $\widehat{P}^{(3)}$ defined in this way is a completely symmetric function of x_5, x_6, x_7 .

Then, we apply (4.3) and (4.2) to define the integrand in (4.6) at three loops:

$$\mathcal{I}_3 = -\frac{1}{(4\pi^2)^3} \frac{x_{13}^4}{\prod_{i=5,6,7} x_{1i}^4 x_{3i}^4} \left[\frac{\widehat{P}_{5,6,7}}{x_{56}^2 x_{67}^2 x_{75}^2} - 6x_{13}^4 \left(\frac{\widehat{P}_{5,6}}{x_{56}^2} + \frac{\widehat{P}_{6,7}}{x_{67}^2} + \frac{\widehat{P}_{5,7}}{x_{57}^2} \right) + 72x_{13}^8 \right], \quad (4.12)$$

where $\widehat{P}_{i,j}$ was defined in (4.7) and the notation was introduced for $\widehat{P}_{5,6,7} \equiv \widehat{P}^{(3)}(x_5, x_6, x_7)$. Replacing the \widehat{P} -polynomials by their explicit expressions we get

$$\begin{aligned} \mathcal{I}_3 = & -\frac{1}{(4\pi^2)^3} \frac{2x_{13}^6}{x_{15}^4 x_{16}^4 x_{17}^4 x_{35}^4 x_{36}^4 x_{37}^4 x_{56}^2 x_{57}^2 x_{67}^2} \left[3x_{13}^6 x_{56}^2 x_{57}^2 x_{67}^2 + x_{13}^4 x_{17}^2 x_{56}^2 (x_{37}^2 x_{56}^2 - 10x_{36}^2 x_{57}^2) \right. \\ & \left. + x_{13}^2 x_{56}^2 (x_{35}^2 x_{36}^2 x_{17}^4 + 2x_{16}^2 x_{35}^2 x_{37}^2 x_{17}^2 + x_{15}^2 x_{16}^2 x_{37}^4) + 4x_{16}^2 x_{17}^4 x_{35}^2 x_{36}^2 + S_3 \text{ perm.} \right]. \end{aligned} \quad (4.13)$$

We verify that the numerator \mathcal{I}_3 vanishes when one of the integration points x_5, x_6, x_7 approaches the external points x_1 or x_3 . This ensures that the integral in (4.6) develops a single pole only. It originates from two different integration regions, $x_5, x_6, x_7 \rightarrow x_1$ and $x_5, x_6, x_7 \rightarrow x_3$, which produce however the same contribution in virtue of the symmetry of \mathcal{I}_3 under the exchange of x_1 and x_3 .

We examine \mathcal{I}_3 for $x_5, x_6, x_7 \rightarrow x_1$ and determine the corresponding function (3.29) at three loops

$$\begin{aligned} \widehat{\mathcal{I}}_3 = & -\frac{1}{(4\pi^2)^3} \frac{2}{x_{15}^4 x_{16}^4 x_{17}^4 x_{56}^2 x_{57}^2 x_{67}^2} \left[4x_{15}^4 x_{16}^2 + x_{15}^4 x_{67}^2 + x_{15}^2 x_{67}^4 + 2x_{15}^2 x_{16}^2 x_{67}^2 + x_{15}^2 x_{16}^2 x_{56}^2 \right. \\ & \left. - 10x_{15}^2 x_{56}^2 x_{67}^2 + 3x_{56}^2 x_{57}^2 x_{67}^2 + S_3 \text{ permutations} \right]. \end{aligned} \quad (4.14)$$

As before, $\widehat{\mathcal{I}}_3$ is a symmetric function of x_5, x_6, x_7 and it does not depend on x_3 . At the next step, we substitute the function $\widehat{\mathcal{I}}_3$ into the left-hand side of (3.30) and integrate it over x_6, x_7 with x_5 fixed

$$\int d^{4-2\epsilon} x_6 d^{4-2\epsilon} x_7 \widehat{\mathcal{I}}_3(x_1 | x_5, x_6, x_7) = \frac{C_2}{2\pi^2} (x_{15}^2)^{-2-2\epsilon}. \quad (4.15)$$

In close analogy with the situation at two loops (see Eq. (3.25) and the discussion around it), the integral on the left-hand side of (4.15) is finite for $\epsilon \rightarrow 0$, due to the special properties of the expression in the square brackets in (4.14). We would like to emphasize that this only holds for the sum of all terms in the square brackets, but not for each individual term. In other words, if we split the integral on the left-hand side of (4.15) into a sum of integrals corresponding to

each terms in (4.14), then each integral develops a pole $1/\epsilon$. The poles cancel in the sum of all integrals leading to a finite expression for the coefficient C_2 in (4.15).⁹ As an example, let us examine the integrals corresponding to the two terms in the second line of (4.14):

$$\begin{aligned} & \int \frac{d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7}{x_{15}^4 x_{16}^4 x_{17}^4 x_{56}^2 x_{57}^2 x_{67}^2} [-10x_{15}^2 x_{56}^2 x_{67}^2 + 3x_{56}^2 x_{57}^2 x_{67}^2] \\ &= -\frac{10}{x_{15}^2} \int \frac{d^{4-2\epsilon}x_6}{x_{16}^4} \int \frac{d^{4-2\epsilon}x_7}{x_{17}^4 x_{57}^2} + \frac{3}{x_{15}^4} \int \frac{d^{4-2\epsilon}x_6}{x_{16}^4} \int \frac{d^{4-2\epsilon}x_7}{x_{17}^4} = 0, \end{aligned} \quad (4.16)$$

where in the second relation we took into account that the integral over x_6 is scaleless and, therefore, it vanishes in dimensional regularization (see footnote 8).

Examining the contribution to (4.15) from the remaining terms on the right-hand side of (4.14), we find that most of the integrals vanish in the same manner as in (4.16). The remaining non-zero contribution takes the following form:

$$\begin{aligned} C_2 = -\frac{1}{8\pi^4} \int d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7 & \left[\frac{2}{x_{17}^4 x_{56}^2 x_{67}^2} + \frac{2}{x_{16}^4 x_{17}^2 x_{56}^2 x_{67}^2} + \frac{2}{x_{16}^2 x_{17}^4 x_{56}^2 x_{67}^2} + \frac{2}{x_{16}^2 x_{17}^2 x_{56}^2 x_{67}^2} \right. \\ & + \frac{1}{x_{16}^4 x_{17}^4 x_{56}^2 x_{57}^2} + \frac{2}{x_{16}^4 x_{17}^2 x_{56}^2 x_{57}^2} + \frac{1}{x_{16}^2 x_{17}^2 x_{56}^2 x_{57}^2} + \frac{4}{x_{16}^4 x_{56}^2 x_{57}^2 x_{67}^2} \\ & \left. + \frac{4}{x_{16}^2 x_{56}^2 x_{57}^2 x_{67}^2} + \frac{4}{x_{16}^4 x_{17}^2 x_{67}^2 x_{57}^2 x_{56}^2} + \frac{x_{67}^2}{x_{16}^4 x_{17}^4 x_{56}^2 x_{57}^2} + \frac{2x_{57}^2}{x_{16}^2 x_{17}^4 x_{56}^2 x_{67}^2} \right]. \end{aligned} \quad (4.17)$$

Here we set $x_{15}^2 = 1$ for simplicity since the dependence of the integral on x_{15}^2 is uniquely fixed by dimension analysis. It is convenient to represent the 12 integrals in this relation in the form of Feynman diagrams as shown in Fig. 2.

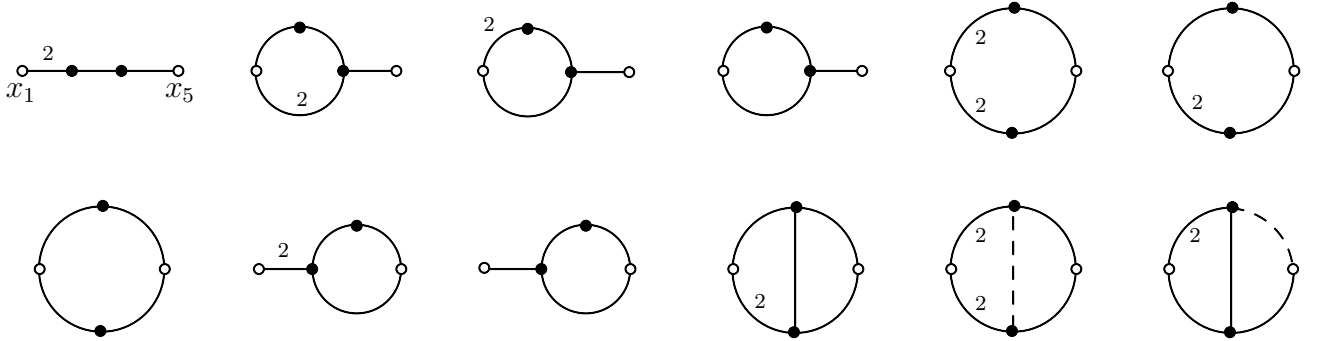


Figure 2: Diagrammatic representation of the integrals in (4.17). Solid lines without labels depict propagators $1/x_{ij}^2$, the label 2 referring to the square of the propagator. Dashed lines represent numerator factors x_{ij}^2 , black and white dots represent integration and external points, respectively.

We observe that the first 9 integrals in (4.17) can be easily evaluated with the help of the “chain relation” (3.26) shown in Fig. 1. Applying (3.26) consecutively, we can express the first 9

⁹To be more precise, the distribution $(x_{15}^2)^{-2-2\epsilon}$ is singular for $\epsilon \rightarrow 0$. So, the left-hand side of (4.15) has a pole $1/\epsilon$ whose residue is a contact term, see Appendix A.

integrals in terms of the G -function and obtain the following relation (with $D = 4 - 2\epsilon$)

$$C_2 = -\frac{1}{8} \left[2G(1, 1)G(2, 2 - D/2) + 2G(1, 1)G(1, 4 - D/2) + 2G(2, 1)G(1, 4 - D/2) \right. \\ \left. + 2G(1, 1)G(1, 3 - D/2) + (G(2, 1))^2 + 2G(1, 1)G(2, 1) + (G(1, 1))^2 \right. \\ \left. + 4G(1, 1)G(2, 3 - D/2) + 4G(1, 1)G(1, 3 - D/2) + I_{10} + I_{11} + I_{12} \right]. \quad (4.18)$$

Here I_{10} , I_{11} and I_{12} stand for the last three integrals on the right-hand side of (4.17), corresponding to the last three diagrams in Fig. 2. The simplest way to compute them is to introduce the dual momenta $k_1 = x_{16}$, $k_2 = x_{17}$ and $p = x_{15}$ and rewrite the above integrals as *two-loop propagator-type* momentum integrals. The latter can be easily computed using MINCER [31], yielding

$$I_{10} = 4(G(1, 1))^2 \left(\frac{1}{2} - \frac{5}{2}\epsilon + \frac{9}{2}\epsilon^2 + O(\epsilon^3) \right), \\ I_{11} = (G(1, 1))^2 (-2 + 4\epsilon - 2\epsilon^2 + O(\epsilon^3)), \\ I_{12} = 2(G(1, 1))^2 (1 - 2\epsilon + 4\epsilon^2 + O(\epsilon^3)). \quad (4.19)$$

Substituting these relations in (4.18) we verify that all poles in ϵ cancel on the right-hand side of (4.18), leading to the Konishi anomalous dimension at three loops, Eq. (3.33),

$$\gamma_{\mathcal{K}}^{(3)} = -C_2 = \frac{21}{4}, \quad (4.20)$$

in agreement with (1.1).

4.3 Four loops

A novel feature that we first encounter at four loops is that the correlation function $F^{(4)}$ receives non-planar corrections

$$F^{(4)} = F_{g=0}^{(4)} + \frac{1}{N_c^2} F_{g=1}^{(4)}. \quad (4.21)$$

The two functions $F_{g=0}^{(4)}$ and $F_{g=1}^{(4)}$ have the same general form (2.5) and (2.6) and are defined by the polynomial

$$P^{(4)}(x_1, \dots, x_8) = P_{g=0}^{(4)} + \frac{1}{N_c^2} P_{g=1}^{(4)}. \quad (4.22)$$

By construction, the polynomials $P_{g=0}^{(4)}$ and $P_{g=1}^{(4)}$ are invariant under S_8 -permutations of the four external points x_1, \dots, x_4 and the four integration points x_5, \dots, x_8 .

Similarly to three loops, the explicit form of the planar polynomial $P_{g=0}^{(4)}$ can be found from the requirement for the correlation function in the planar sector to have the correct asymptotic

behaviour in the light-cone limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \rightarrow 0$. The result is [2]

$$\begin{aligned} P_{g=0}^{(4)} &= \frac{1}{24} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{56}^2 x_{78}^6 \\ &+ \frac{1}{8} x_{12}^2 x_{13}^2 x_{16}^2 x_{24}^2 x_{27}^2 x_{34}^2 x_{38}^2 x_{45}^2 x_{56}^4 x_{78}^4 \\ &- \frac{1}{16} x_{12}^2 x_{15}^2 x_{18}^2 x_{23}^2 x_{26}^2 x_{34}^2 x_{37}^2 x_{45}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{78}^2 + S_8 \text{ permutations}. \end{aligned} \quad (4.23)$$

In the non-planar sector, the same requirement turns out to be less restrictive but it allowed us to determine the non-planar polynomial $P_{g=1}^{(4)}$ up to four arbitrary constants which are expected to take rational values. In order to fix these constants, one would need more detailed information about the properties of the correlation function. To save space, here we do not present the general expression for $P_{g=1}^{(4)}$, it can be found in Ref. [2].

In fact, for our purposes we only need the expressions for the polynomials $P_{g=0}^{(4)}$ and $P_{g=1}^{(4)}$ in the short-distance limit $x_2 \rightarrow x_1$ and $x_4 \rightarrow x_3$, Eq. (4.4). For the planar polynomial we find from (4.23) for $x_2 = x_1$ and $x_4 = x_3$

$$\begin{aligned} \widehat{P}_{g=0}^{(4)} &= x_{13}^8 (x_{15}^2 x_{18}^2 x_{36}^2 x_{37}^2 x_{58}^4 x_{67}^4 + x_{16}^2 x_{17}^2 x_{36}^2 x_{37}^2 x_{58}^6 x_{67}^2 + 4x_{17}^2 x_{18}^2 x_{36}^2 x_{37}^2 x_{56}^4 x_{58}^2 x_{67}^2 \\ &+ 4x_{16}^2 x_{17}^2 x_{37}^2 x_{38}^2 x_{56}^4 x_{58}^2 x_{67}^2 + 2x_{17}^2 x_{18}^2 x_{37}^2 x_{38}^2 x_{56}^4 x_{58}^2 x_{67}^2 + 4x_{17}^2 x_{18}^2 x_{36}^2 x_{38}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 \\ &+ 2x_{16}^2 x_{18}^2 x_{35}^2 x_{37}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{13}^8) + 2x_{13}^6 (x_{17}^2 x_{18}^4 x_{37}^2 x_{38}^2 x_{56}^6 + 2x_{17}^4 x_{18}^2 x_{36}^2 x_{38}^4 x_{56}^4 x_{57}^2 \\ &+ 2x_{16}^2 x_{18}^4 x_{37}^2 x_{38}^2 x_{56}^4 x_{57}^2 + 2x_{17}^2 x_{18}^4 x_{36}^2 x_{37}^2 x_{56}^2 x_{57}^2 x_{58}^2 + 2x_{15}^2 x_{18}^4 x_{36}^2 x_{37}^2 x_{56}^4 x_{58}^2 x_{67}^2) \\ &+ 2x_{13}^4 (x_{17}^2 x_{18}^6 x_{35}^2 x_{36}^2 x_{37}^4 x_{56}^4 + x_{15}^2 x_{16}^2 x_{18}^4 x_{37}^2 x_{38}^2 x_{56}^4 + 2x_{16}^2 x_{17}^2 x_{18}^4 x_{35}^2 x_{37}^2 x_{38}^2 x_{56}^4 \\ &+ 2x_{15}^2 x_{16}^2 x_{17}^2 x_{36}^2 x_{38}^2 x_{56}^2 x_{57}^2 + 2x_{15}^2 x_{17}^4 x_{18}^2 x_{36}^2 x_{38}^2 x_{56}^2 x_{57}^2 + 2x_{17}^2 x_{18}^6 x_{35}^2 x_{36}^2 x_{37}^2 x_{56}^2 x_{57}^2 \\ &+ 2x_{17}^4 x_{18}^4 x_{35}^2 x_{36}^2 x_{38}^2 x_{56}^2 x_{57}^2 - x_{16}^2 x_{17}^2 x_{18}^4 x_{35}^2 x_{36}^2 x_{37}^2 x_{58}^2 x_{67}^2) + 4x_{13}^2 (x_{15}^2 x_{16}^2 x_{18}^6 x_{35}^2 x_{36}^2 x_{37}^2 x_{56}^2 \\ &+ 2x_{16}^2 x_{17}^2 x_{18}^6 x_{35}^2 x_{36}^2 x_{37}^2 x_{56}^2 + 2x_{15}^2 x_{16}^4 x_{18}^2 x_{35}^2 x_{37}^2 x_{38}^2 x_{56}^2 + 2x_{16}^2 x_{17}^2 x_{18}^4 x_{35}^2 x_{37}^2 x_{38}^2 x_{56}^2 \\ &- x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^4 x_{35}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{56}^2) + S_4 \text{ permutation}. \end{aligned} \quad (4.24)$$

Here the S_4 permutations are needed to restore the symmetry of $\widehat{P}_{g=0}^{(4)}$ under the exchange of the integration points x_5, \dots, x_8 . The polynomial $\widehat{P}_{g=0}^{(4)}$ is also invariant under the exchange of the external points x_1 and x_3 , so that it has an $S_2 \times S_4$ permutation symmetry. Notice that the relatively long expression for $\widehat{P}_{g=0}^{(4)}$, as compared with (4.23), is an artifact of decomposing the S_8 permutations into $S_4 \times S_4$ ones.

For the non-planar polynomial the situation is just the opposite. The expression for $P_{g=1}^{(4)}$ is very lengthy whereas in the short-distance limit it takes a remarkably simple form. Namely, the four different polynomials that accompany the four arbitrary constants in the expression for $P_{g=1}^{(4)}$ become proportional to each other at $x_2 = x_1$ and $x_4 = x_3$, leading to [2]

$$\widehat{P}_{g=1}^{(4)} = \lim_{\substack{x_1 \rightarrow x_2 \\ x_3 \rightarrow x_4}} P_{g=1}^{(4)} = \frac{1}{6} c_{g=1}^{(4)} x_{13}^4 (x_{56}^2 x_{78}^2 + x_{57}^2 x_{68}^2 + x_{58}^2 x_{67}^2) \prod_{i=5,6,7,8} x_{1i}^2 x_{3i}^2, \quad (4.25)$$

with $c_{g=1}^{(4)}$ given by a linear combination of the four rational constants mentioned above.

Then, we substitute the polynomial $\widehat{P}^{(4)}$ into (4.3) and use (4.2) to obtain the expression for $I^{(4)}$. Going to the corresponding integrand \mathcal{I}_4 , Eq. (4.6), we find

$$\mathcal{I}_4 = \mathcal{I}_{4,g=0} + \frac{1}{N_c^2} \mathcal{I}_{4,g=1}. \quad (4.26)$$

Here, the expression for the non-planar correction reads

$$\begin{aligned}\mathcal{I}_{4,g=1} &= \frac{6}{4!(4\pi^2)^4} \frac{x_{13}^4}{(x_{15}^4 x_{35}^4) \dots (x_{18}^4 x_{38}^4)} \frac{\widehat{P}_{g=1}^{(4)}(x_5, \dots, x_8)}{\prod_{5 \leq i < j \leq 8} x_{ij}^2} \\ &= \frac{c_{g=1}^{(4)}}{4!(4\pi^2)^4} \frac{x_{13}^8}{(x_{15}^2 x_{35}^2) \dots (x_{18}^2 x_{38}^2)} \frac{(x_{56}^2 x_{78}^2 + x_{57}^2 x_{68}^2 + x_{58}^2 x_{67}^2)}{x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2},\end{aligned}\quad (4.27)$$

where in the second relation we replaced $\widehat{P}_{g=1}^{(4)}$ by its explicit expression (4.25). In the planar sector, the analogous expression for $\mathcal{I}_{4,g=0}$ is much longer since it involves the \widehat{P} -polynomials at lower loops:

$$\begin{aligned}\mathcal{I}_{4,g=0} &= \frac{6}{4!(4\pi^2)^4} \frac{x_{13}^4}{\prod_{i=5}^8 x_{1i}^4 x_{3i}^4} \left[\frac{1}{4!} \frac{\widehat{P}_{5,6,7,8}}{x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} - x_{13}^4 \frac{\widehat{P}_{5,6,7}}{x_{56}^2 x_{57}^2 x_{67}^2} \right. \\ &\quad \left. - \frac{3}{4} x_{13}^4 \frac{\widehat{P}_{5,6} \widehat{P}_{7,8}}{x_{56}^2 x_{78}^2} + 18(x_{13}^4)^2 \frac{\widehat{P}_{5,6}}{x_{56}^2} - 54(x_{13}^4)^3 \right] + S_4 \text{ permutations}\end{aligned}\quad (4.28)$$

where $\widehat{P}_{5,6,7,8} \equiv \widehat{P}_{g=0}^{(4)}$ is given by (4.24) and the right-hand side is symmetrized with respect to all S_4 permutations of the integration points x_5, \dots, x_8 . Replacing the \widehat{P} -polynomials by their explicit expressions, we obtain a lengthy result for $\mathcal{I}_{4,g=0}$.

At the next step, we restrict all the integration points x_5, \dots, x_8 to the vicinity of the external point x_1 and simplify the integrand \mathcal{I}_4 by replacing $x_{i3}^2 \rightarrow x_{13}^2$ (with $i = 5, \dots, 8$). Denoting the resulting function $\widehat{\mathcal{I}}_4$, we find from (4.27) and (4.28)

$$\widehat{\mathcal{I}}_{4,g=1} = \frac{c_{g=1}^{(4)}}{4!(4\pi^2)^4} \frac{x_{56}^2 x_{78}^2 + x_{57}^2 x_{68}^2 + x_{58}^2 x_{67}^2}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} \quad (4.29)$$

in the non-planar sector, and

$$\begin{aligned}\widehat{\mathcal{I}}_{4,g=0} &= \frac{6}{4!(4\pi^2)^4} \frac{1}{x_{15}^4 x_{16}^4 x_{17}^4 x_{18}^4 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} \\ &\quad \times \left[2x_{17}^2 x_{56}^4 x_{18}^6 + 4x_{15}^2 x_{16}^2 x_{56}^2 x_{18}^6 + 8x_{16}^2 x_{17}^2 x_{56}^2 x_{18}^6 + 4x_{17}^2 x_{56}^2 x_{57}^2 x_{18}^6 + 2x_{17}^2 x_{56}^6 x_{18}^4 \right. \\ &\quad + 2x_{15}^2 x_{16}^2 x_{56}^4 x_{18}^4 + 4x_{16}^2 x_{17}^2 x_{56}^4 x_{18}^4 + 8x_{15}^2 x_{16}^4 x_{56}^2 x_{18}^4 + 8x_{16}^4 x_{17}^2 x_{56}^2 x_{18}^4 \\ &\quad - 4x_{15}^2 x_{16}^2 x_{17}^2 x_{56}^4 x_{18}^4 + 4x_{16}^2 x_{56}^4 x_{57}^2 x_{18}^4 + 4x_{17}^2 x_{56}^2 x_{57}^2 x_{18}^4 - 44x_{17}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{18}^4 \\ &\quad - 2x_{16}^2 x_{17}^2 x_{58}^2 x_{67}^2 x_{18}^4 + 4x_{15}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{18}^4 - 12x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{18}^4 + x_{15}^2 x_{58}^4 x_{67}^2 x_{18}^4 \\ &\quad - 12x_{56}^2 x_{57}^2 x_{58}^4 x_{67}^2 x_{18}^4 + 4x_{17}^2 x_{56}^4 x_{57}^2 x_{18}^4 + 4x_{15}^2 x_{17}^2 x_{56}^2 x_{57}^2 x_{18}^4 + 4x_{17}^2 x_{56}^2 x_{58}^4 x_{67}^2 x_{18}^4 \\ &\quad + 2x_{17}^2 x_{56}^4 x_{58}^2 x_{67}^2 x_{18}^2 - 20x_{17}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{18}^2 - 46x_{16}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{18}^2 \\ &\quad + 96x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{18}^2 + 4x_{15}^2 x_{16}^2 x_{17}^2 x_{56}^2 x_{57}^2 + x_{16}^2 x_{17}^2 x_{58}^6 x_{67}^2 + 4x_{16}^2 x_{17}^2 x_{56}^4 x_{58}^2 x_{67}^2 \\ &\quad \left. - 12x_{16}^2 x_{17}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 - 21x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2 \right] + S_4 \text{ permutation}\end{aligned}\quad (4.30)$$

in the planar sector. We verify that the functions $\widehat{\mathcal{I}}_{4,g=0}$ and $\widehat{\mathcal{I}}_{4,g=1}$ transform covariantly under conformal transformations and do not depend on the external point x_3 .

Then, to find the four-loop correction to the Konishi anomalous dimension (3.33), or equivalently the coefficient C_3 , we substitute $\widehat{\mathcal{I}}_4$ into (3.30) and obtain:

$$C_3 = \frac{2\pi^2}{(x_{15}^2)^{2+3\epsilon}} \int d^{4-2\epsilon} x_6 d^{4-2\epsilon} x_7 d^{4-2\epsilon} x_8 \left(\widehat{\mathcal{I}}_{4,g=0} + \frac{1}{N_c^2} \widehat{\mathcal{I}}_{4,g=1} \right). \quad (4.31)$$

By construction, C_3 is dimensionless and is expected to be finite as $\epsilon \rightarrow 0$. We do not remove the regularization, however, since it is more advantageous to expand the right-hand side of (4.31) into a sum of basis three-loop (divergent) integrals and evaluate each of them separately.

Let us start with the non-planar contribution to (4.31). Setting for simplicity $x_{15}^2 = 1$, we find from (4.29), (4.31)

$$C_{3,g=1} = \frac{c_{g=1}^{(4)}}{1024\pi^6} \int \frac{d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7 d^{4-2\epsilon}x_8}{x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{78}^2}. \quad (4.32)$$

The integral on the right-hand side of this relation is shown diagrammatically in Fig. 3. It is finite at $\epsilon = 0$ and, most importantly, it corresponds to a planar graph. This fact allows us to introduce dual momenta and represent the integral in the form of a planar dual (momentum) graph, as shown in Fig. 3. The main advantage of using the dual representation is that the

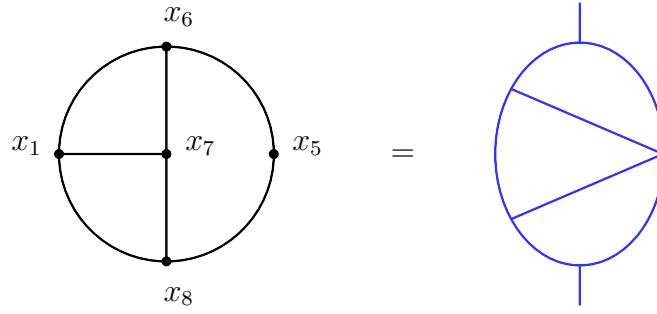


Figure 3: Diagrammatic representation of the integral (4.32) in the x -representation (left) and in the dual momentum representation (right). In what follows, all momentum integrals are shown in blue.

resulting three-loop momentum integral can be easily evaluated using the **MINCER** package [31], leading to

$$C_{3,g=1} = \frac{c_{g=1}^{(4)}}{1024} \times (20\zeta_5 + O(\epsilon)). \quad (4.33)$$

We recall that the constant $c_{g=1}^{(4)}$ is expected to take rational values.

Let us now examine the three-loop integrals in (4.31) generated by the integrand $\widehat{\mathcal{I}}_{4,g=0}$, Eq. (4.30). We shall denote the corresponding contribution to the right-hand side of (4.31) by $C_{3,g=0}$. The three-loop propagator integrals appearing in the calculation belong to the following family of three-fold integrals, with various integer indices a_1, \dots, a_9

$$G(a_1, \dots, a_9) = \int \frac{d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7 d^{4-2\epsilon}x_8}{(x_{16}^2)^{a_1} (x_{17}^2)^{a_2} (x_{18}^2)^{a_3} (x_6^2)^{a_4} (x_7^2)^{a_5} (x_8^2)^{a_6} (x_{67}^2)^{a_7} (x_{68}^2)^{a_8} (x_{78}^2)^{a_9}}, \quad (4.34)$$

where we put $x_5 = 0$ for simplicity. Notice that the indices a_i can take both positive and negative values. In the latter case, the corresponding term appears in the numerator. For arbitrary choices of the indices a_i some of the integrals (4.34) may be non-planar and so can not be rewritten as dual momentum integrals of propagator type. However, most importantly, in our case all the integrals appearing in (4.31) are planar and thus of propagator type.

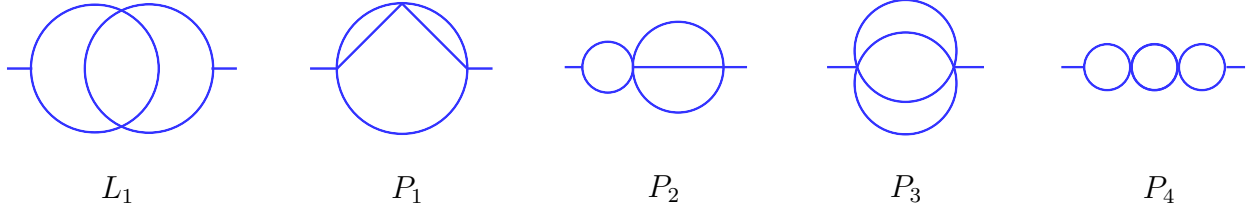


Figure 4: The dual momentum master integrals defining the four-loop Konishi anomalous dimension.

To evaluate each of them we apply the standard technique called integration by parts (IBP) [33] (see Chapter 5 of [34] for a review of the method) which provides the possibility of representing a given integral of this family as a linear combination of so-called master integrals. We found that the calculation of $C_{3,g=0}$ involves only 5 master integrals, all corresponding to planar graphs. Therefore, introducing the dual momenta $k_i = x_i - x_{i+1}$, we can rewrite these integrals as the dual momentum-space integrals L_1, P_1, \dots, P_4 shown in Fig. 4.

The resulting expression for $C_{3,g=0}$ is

$$\begin{aligned}
C_{3,g=0} = & L_1 \left(\frac{9}{64\epsilon^2} - \frac{3}{4\epsilon} + \frac{45}{16} - \frac{165\epsilon}{16} + \frac{1647\epsilon^2}{64} - 54\epsilon^3 + O(\epsilon^4) \right) \\
& + P_1 \left(-\frac{9}{32\epsilon^3} + \frac{93}{32\epsilon^2} - \frac{981}{64\epsilon} + \frac{2967}{64} - \frac{4035\epsilon}{32} + \frac{2961\epsilon^2}{8} + O(\epsilon^3) \right) \\
& + P_2 \left(\frac{9}{16\epsilon^3} - \frac{99}{32\epsilon^2} + \frac{195}{16\epsilon} - \frac{1815}{32} + \frac{1749\epsilon}{8} - \frac{9585\epsilon^2}{16} + O(\epsilon^3) \right) \\
& + P_3 \left(\frac{27}{4\epsilon^3} - \frac{1035}{16\epsilon^2} + \frac{3267}{32\epsilon} + \frac{8415}{16} - \frac{62247\epsilon}{32} + O(\epsilon^2) \right) \\
& + P_4 \left(\frac{3}{16\epsilon} - \frac{45}{64} + \frac{63\epsilon}{64} - \frac{9\epsilon^2}{32} + O(\epsilon^4) \right). \tag{4.35}
\end{aligned}$$

The three-loop massless propagator master integrals in this relation were evaluated many years ago [33].¹⁰ Using the known results [27] for the master three-loop integrals L_1, P_1, \dots, P_4 we find from (4.35)

$$C_{3,g=0} = \frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5 + O(\epsilon). \tag{4.36}$$

Substituting (4.33) and (4.36) into (3.33) we finally obtain the following result for the four-loop correction to the Konishi anomalous dimension

$$\gamma_{\mathcal{K}}^{(4)} = -\frac{39}{4} + \frac{9}{4}\zeta_3 - \frac{45}{8}\zeta_5 + \frac{r}{N_c^2}\zeta_5, \tag{4.37}$$

where $r = -5c_{g=1}^{(4)}/256$ is an undetermined rational constant. The planar correction to $\gamma_{\mathcal{K}}^{(4)}$ is in agreement with the known result [5, 6, 7, 15, 16, 17]. The non-planar correction has been

¹⁰All the integrals appearing in (4.31) can be handled by MINCER [31]. In addition to the single integral (4.32) contributing to the non-planar correction to the anomalous dimension, we found 76 integrals in the planar sector. Evaluating them with the help of MINCER, we arrived at the same result (4.36). However, we would like to emphasize that at five loops the reduction of more than 17000 integrals to master integrals can only be done by a direct application of the IBP method (see Sect. 5).

computed in [26], and the result confirms our prediction. Moreover, it allows us to fix the value of the unknown constant in (4.37):

$$r = -\frac{135}{2}. \quad (4.38)$$

In this section we have demonstrated the high efficiency of our method for computing the Konishi anomalous dimension at four loops. We emphasize once again that we do not use the conventional Feynman diagram technique. For comparison, the direct calculation of [15, 16] involves hundreds of $\mathcal{N} = 1$ super-graphs, each giving rise to a number of Feynman integrals; in the calculation of [17] the number of contributing Feynman graphs exceeds 130 000.

5 Konishi anomalous dimension at five loops

It is straightforward to extend our analysis to five loops. In this case, the correlation function $F^{(5)}$ has the following form

$$F^{(5)}(x_i) = F_{g=0}^{(5)} + \frac{1}{N_c^2} F_{g=1}^{(5)} + \frac{1}{N_c^4} F_{g=2}^{(5)}. \quad (5.1)$$

In what follows, we shall restrict the discussion to the planar sector only. As was shown in Ref. [2], the five-loop correlation function in the planar limit is given by

$$F_{g=0}^{(5)}(x_i) = \frac{1}{5!(-4\pi^2)^5} \int \frac{d^4x_5 d^4x_6 d^4x_7 d^4x_8 d^4x_9 P^{(5)}(x_1, \dots, x_9)}{x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2 \prod_{i=1}^4 x_{i5}^2 x_{i6}^2 x_{i7}^2 x_{i8}^2 x_{i9}^2}, \quad (5.2)$$

where the polynomial $P^{(5)}$ is invariant under the S_9 permutations of the four external points x_1, \dots, x_4 and the five integration points x_5, \dots, x_9 . It is given by the following expression:

$$\begin{aligned} P^{(5)} = & -\frac{1}{2}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^2x_{26}^2x_{29}^2x_{37}^2x_{38}^2x_{39}^2x_{47}^2x_{48}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2, \\ & +\frac{1}{4}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^2x_{26}^2x_{29}^2x_{37}^2x_{39}^2x_{48}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\ & +\frac{1}{4}x_{13}^2x_{17}^2x_{19}^2x_{24}^2x_{26}^2x_{27}^2x_{29}^2x_{36}^2x_{39}^2x_{48}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\ & +\frac{1}{6}x_{13}^2x_{16}^2x_{19}^2x_{24}^2x_{28}^2x_{29}^2x_{37}^2x_{38}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{68}^2 \\ & -\frac{1}{8}x_{13}^2x_{16}^2x_{18}^2x_{24}^2x_{28}^2x_{29}^2x_{37}^2x_{39}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{69}^2x_{78}^2 \\ & +\frac{1}{28}x_{13}^2x_{17}^2x_{18}^2x_{19}^2x_{24}^2x_{36}^2x_{38}^2x_{39}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2x_{69}^2x_{78}^2 \\ & +\frac{1}{12}x_{13}^2x_{16}^2x_{17}^2x_{19}^2x_{26}^2x_{27}^2x_{28}^2x_{29}^2x_{35}^2x_{38}^2x_{39}^2x_{45}^2x_{46}^2x_{47}^2x_{49}^2x_{57}^2x_{58}^2x_{68}^2 + S_9 \text{ permutations}, \end{aligned} \quad (5.3)$$

where the relative coefficients follow from the requirement for the correlation function to have the correct asymptotic behaviour at short distances.

The analysis goes along the same lines as at four loops. We start with examining the integrand (5.2) in the double short-distance limit $x_2 \rightarrow x_1$ and $x_4 \rightarrow x_3$ and, then, apply (4.2) to identify the five-loop integrand of $I^{(5)}$:

$$\begin{aligned} \mathcal{I}_5 = & -\frac{6}{5!(4\pi^2)^5} \frac{x_{13}^4}{\prod_{i=5}^9 x_{1i}^4 x_{3i}^4} \left[\frac{1}{5!} \frac{\hat{P}_{5,6,7,8,9}}{x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} - \frac{1}{4} x_{12}^4 \frac{\hat{P}_{5,6,7,8}}{x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} \right. \\ & - \frac{1}{2} x_{13}^4 \frac{\hat{P}_{5,6,7}}{x_{56}^2 x_{57}^2 x_{67}^2} \frac{\hat{P}_{8,9}}{x_{89}^2} + 6(x_{13}^4)^2 \frac{\hat{P}_{5,6,7}}{x_{56}^2 x_{57}^2 x_{67}^2} + 9(x_{13}^4)^2 \frac{\hat{P}_{5,6}}{x_{56}^2} \frac{\hat{P}_{7,8}}{x_{78}^2} \\ & \left. - 108(x_{13}^4)^3 \frac{\hat{P}_{5,6}}{x_{56}^2} + \frac{1296}{5} (x_{13}^4)^4 \right] + S_5 \text{ permutations}, \end{aligned} \quad (5.4)$$

where $\widehat{P}_{5,6,7,8,9}$ is the polynomial $P^{(5)}$ evaluated at $x_2 = x_1$ and $x_4 = x_3$ and the remaining \widehat{P} -polynomials were defined earlier (see Eqs. (4.7), (4.12) and (4.28)). The expression on the right-hand side of (5.4) is symmetrized with respect to S_5 permutations of the integration points x_5, \dots, x_9 . To save space, here we do not present the explicit expression for \mathcal{I}_5 .

At the next step, we simplify the expression for \mathcal{I}_5 by choosing all integration points x_5, \dots, x_9 to lie in the vicinity of the point x_1 . This is equivalent to sending the external point to infinity $x_3 \rightarrow \infty$ with all remaining points fixed

$$\widehat{\mathcal{I}}_5 = \lim_{x_3 \rightarrow \infty} \mathcal{I}_5(x_1, x_3; x_5, \dots, x_9). \quad (5.5)$$

Finally, we apply (3.33) to express the coefficient C_4 as the following four-loop integral

$$C_4 = \frac{2\pi^2}{(x_{15}^2)^{2+4\epsilon}} \int d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7 d^{4-2\epsilon}x_8 d^{4-2\epsilon}x_9 \widehat{\mathcal{I}}_5(x_1; x_5, \dots, x_9). \quad (5.6)$$

As before, to simplify the calculation we put $x_5 = 0$.

Replacing $\widehat{\mathcal{I}}_5$ in (5.6) by its explicit expression, we find that C_4 is given by the sum of more than 17000 four-loop two-point Feynman integrals. All of them belong to the following family of four-fold integrals, with various integer (positive and negative) indices a_1, \dots, a_{14}

$$G(a_1, \dots, a_{14}) = \int \frac{d^{4-2\epsilon}x_6 d^{4-2\epsilon}x_7 d^{4-2\epsilon}x_8 d^{4-2\epsilon}x_9}{(x_{16}^2)^{a_1} (x_{17}^2)^{a_2} (x_{18}^2)^{a_3} (x_{19}^2)^{a_4} (x_6^2)^{a_5} (x_7^2)^{a_6} (x_8^2)^{a_7}} \times \frac{1}{(x_9^2)^{a_8} (x_{67}^2)^{a_9} (x_{68}^2)^{a_{10}} (x_{69}^2)^{a_{11}} (x_{78}^2)^{a_{12}} (x_{79}^2)^{a_{13}} (x_{89}^2)^{a_{14}}}. \quad (5.7)$$

As in the four-loop case, to evaluate each of them we apply the IBP method [33]. To solve the IBP relations, i.e. to represent every integral on the right-hand side of (5.6) as a linear combination of master integrals, we apply the C++ version of the code FIRE [35]. In this way, we found that C_4 is given by a linear combination of 22 master integrals:

$$C_4 = w_{44}M_{44} + w_{61}M_{61} + w_{36}M_{36} + w_{31}M_{31} + w_{35}M_{35} + w_{22}M_{22} + w_{32}M_{32} \\ + w_{33}M_{33} + w_{34}M_{34} + w_{25}M_{25} + w_{23}M_{23} + w_{27}M_{27} + w_{24}M_{24} + w_{26}M_{26} \\ + w_{01}M_{01} + w_{21}M_{21} + w_{12}M_{12} + w_{11}M_{11} + w_{14}M_{14} + w_{13}M_{13} + w_1 I_1 + w_2 I_2, \quad (5.8)$$

with the coefficient functions w_i defined below in Eq. (5.9). Among the master integrals only two, I_1 and I_2 , are associated with non-planar graphs (see Eqs. (B.1) below). The remaining 20 master integrals M_{44}, \dots, M_{13} correspond to planar graphs. This allows us to introduce the dual momenta $k_i = x_i - x_{i+1}$ and represent the same integrals as four-loop propagator master (momentum) integrals shown in Fig. 5. The latter integrals were calculated recently in [27] as an ϵ expansion up to transcendentality weight seven.¹¹ The explicit expressions for the planar integrals M_{44}, \dots, M_{13} can be found in [27]. To save space, we do not present them here.

¹¹At the moment, results for the master integrals are known up to transcendentality weight twelve [36].

The corresponding coefficient functions are given by

$$\begin{aligned}
w_{44} &= -\frac{3}{80} + \dots, \\
w_{61} &= \frac{3}{128} + \frac{3}{32}\epsilon + \dots, \\
w_{36} &= \frac{1}{10}\epsilon^{-1} - \frac{189}{320} + \frac{11}{16}\epsilon + \dots, \\
w_{31} &= -\frac{3}{64}\epsilon^{-2} + \frac{81}{320}\epsilon^{-1} - \frac{27}{40} + \frac{57}{80}\epsilon - \frac{9}{40}\epsilon^2 + \dots, \\
w_{35} &= -\frac{3}{80}\epsilon^{-2} + \frac{233}{480}\epsilon^{-1} - \frac{189}{80} + \frac{2131}{480}\epsilon - \frac{509}{80}\epsilon^2 + \dots, \\
w_{22} &= -\frac{11}{20}\epsilon^{-3} + \frac{295}{64}\epsilon^{-2} - \frac{85}{32}\epsilon^{-1} - \frac{39711}{640} + \frac{349167}{1280}\epsilon - \frac{358191}{512}\epsilon^2 + \frac{1748923}{1024}\epsilon^3 + \dots, \\
w_{32} &= -\frac{3}{80}\epsilon^{-3} + \frac{9}{32}\epsilon^{-2} - \frac{261}{160}\epsilon^{-1} + \frac{1281}{160} - \frac{1953}{64}\epsilon + \frac{3879}{40}\epsilon^2 - \frac{2313}{8}\epsilon^3 + \frac{1701}{2}\epsilon^4 + \dots, \\
w_{33} &= -\frac{7}{80}\epsilon^{-3} + \frac{57}{64}\epsilon^{-2} - \frac{179}{40}\epsilon^{-1} + \frac{4911}{320} - \frac{6269}{160}\epsilon + \frac{1723}{20}\epsilon^2 - \frac{2631}{16}\epsilon^3 + \frac{17481}{80}\epsilon^4 + \dots, \\
w_{34} &= -\frac{27}{320}\epsilon^{-3} + \frac{21}{20}\epsilon^{-2} - \frac{3629}{640}\epsilon^{-1} + \frac{1141}{64} - \frac{5643}{128}\epsilon + \frac{475}{4}\epsilon^2 - \frac{25953}{80}\epsilon^3 + \frac{8817}{10}\epsilon^4 + \dots, \\
w_{25} &= \frac{9}{80}\epsilon^{-4} - \frac{313}{160}\epsilon^{-3} + \frac{4221}{320}\epsilon^{-2} - \frac{25583}{640}\epsilon^{-1} + \frac{42073}{640} - \frac{3341}{32}\epsilon + \frac{12279}{64}\epsilon^2 - \frac{4131}{32}\epsilon^3 + \dots, \\
w_{23} &= -\frac{3}{80}\epsilon^{-4} - \frac{3}{10}\epsilon^{-3} + \frac{81}{80}\epsilon^{-2} + \frac{9939}{640}\epsilon^{-1} - \frac{83889}{640} + \frac{183273}{320}\epsilon - \frac{622629}{320}\epsilon^2 + \frac{978141}{160}\epsilon^3 + \dots, \\
w_{27} &= \frac{21}{160}\epsilon^{-4} - \frac{753}{320}\epsilon^{-3} + \frac{5319}{320}\epsilon^{-2} - \frac{9963}{160}\epsilon^{-1} + \frac{27913}{160} - \frac{42339}{80}\epsilon + \frac{139719}{80}\epsilon^2 - \frac{448299}{80}\epsilon^3 + \dots, \\
w_{24} &= -\frac{3}{20}\epsilon^{-4} + \frac{439}{160}\epsilon^{-3} - \frac{5141}{320}\epsilon^{-2} + \frac{20549}{640}\epsilon^{-1} + \frac{16623}{1280} - \frac{630099}{2560}\epsilon + \frac{5544039}{5120}\epsilon^2 - \frac{8490959}{2048}\epsilon^3 + \dots, \\
w_{26} &= \frac{19}{80}\epsilon^{-4} - \frac{1329}{320}\epsilon^{-3} + \frac{127}{4}\epsilon^{-2} - \frac{43443}{320}\epsilon^{-1} + \frac{504477}{1280} - \frac{2516491}{2560}\epsilon + \frac{10666651}{5120}\epsilon^2 - \frac{22997991}{10240}\epsilon^3 + \dots, \\
w_{01} &= -\frac{213}{4}\epsilon^{-5} + \frac{25311}{40}\epsilon^{-4} - \frac{1038647}{960}\epsilon^{-3} - \frac{8046763}{480}\epsilon^{-2} + \frac{38217863}{320}\epsilon^{-1} - \frac{398050007}{960} + \frac{2071600273}{1920}\epsilon + \dots, \\
w_{21} &= -\frac{133}{160}\epsilon^{-4} + \frac{61}{6}\epsilon^{-3} - \frac{30023}{480}\epsilon^{-2} + \frac{84321}{320}\epsilon^{-1} - \frac{329747}{384} + \frac{9067319}{3840}\epsilon - \frac{14362073}{2560}\epsilon^2 + \frac{59375477}{5120}\epsilon^3 + \dots, \\
w_{12} &= -\frac{61}{64}\epsilon^{-5} + \frac{2149}{160}\epsilon^{-4} - \frac{122841}{1280}\epsilon^{-3} + \frac{576611}{1280}\epsilon^{-2} - \frac{2066537}{1280}\epsilon^{-1} + \frac{6439581}{1280} - \frac{1803913}{128}\epsilon + \frac{11613069}{320}\epsilon^2 + \dots, \\
w_{11} &= -\frac{471}{160}\epsilon^{-5} + \frac{15583}{320}\epsilon^{-4} - \frac{346883}{960}\epsilon^{-3} + \frac{509833}{320}\epsilon^{-2} - \frac{5155033}{960}\epsilon^{-1} + \frac{1322649}{80} - \frac{1855317}{40}\epsilon + \frac{9142761}{80}\epsilon^2 + \dots, \\
w_{14} &= -\frac{51}{160}\epsilon^{-5} + \frac{361}{64}\epsilon^{-4} - \frac{4829}{80}\epsilon^{-3} + \frac{471857}{960}\epsilon^{-2} - \frac{677747}{240}\epsilon^{-1} + \frac{14881773}{1280} - \frac{289328273}{7680}\epsilon + \frac{1662945973}{15360}\epsilon^2 + \dots, \\
w_{13} &= \frac{573}{160}\epsilon^{-5} - \frac{5431}{80}\epsilon^{-4} + \frac{1084379}{1920}\epsilon^{-3} - \frac{1648533}{640}\epsilon^{-2} + \frac{16101181}{1920}\epsilon^{-1} - \frac{16307253}{640} + \frac{12145053}{160}\epsilon - \frac{65454483}{320}\epsilon^2 + \dots
\end{aligned} \tag{5.9}$$

Here the series expansion of w_i is truncated at the order in ϵ which is related to the maximal power of $1/\epsilon$ in the expression for the corresponding basis integral M_i in (5.8), so that the right-hand side of (5.8) can be evaluated at order $O(\epsilon^0)$.

The two non-planar master integrals I_1 and I_2 entering the right-hand side of (5.8) are shown diagrammatically in Fig. 6 and their explicit form can be found in (B.1). These integrals are evaluated in Appendix B leading to

$$\begin{aligned}
I_1 &= \frac{5\zeta_5}{\epsilon} + \frac{5}{378}\pi^6 - 13\zeta_3^2 + 35\zeta_5 + \left(-\frac{13}{30}\pi^4\zeta_3 - 91\zeta_3^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6 \right)\epsilon + \dots \\
I_2 &= -\frac{20\zeta_5}{\epsilon} - \frac{10}{189}\pi^6 - 8\zeta_3^2 - 40\zeta_5 + \left(-\frac{4}{15}\pi^4\zeta_3 - 16\zeta_3^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6 \right)\epsilon + \dots
\end{aligned} \tag{5.10}$$

Their coefficient functions in (5.8) are

$$\begin{aligned}
w_1 &= \frac{3}{80}\epsilon^{-1} - \frac{21}{80} + \frac{741}{640}\epsilon + O(\epsilon^2), \\
w_2 &= \frac{9}{160}\epsilon^{-1} - \frac{9}{80} + \frac{807}{320}\epsilon + O(\epsilon^2).
\end{aligned} \tag{5.11}$$

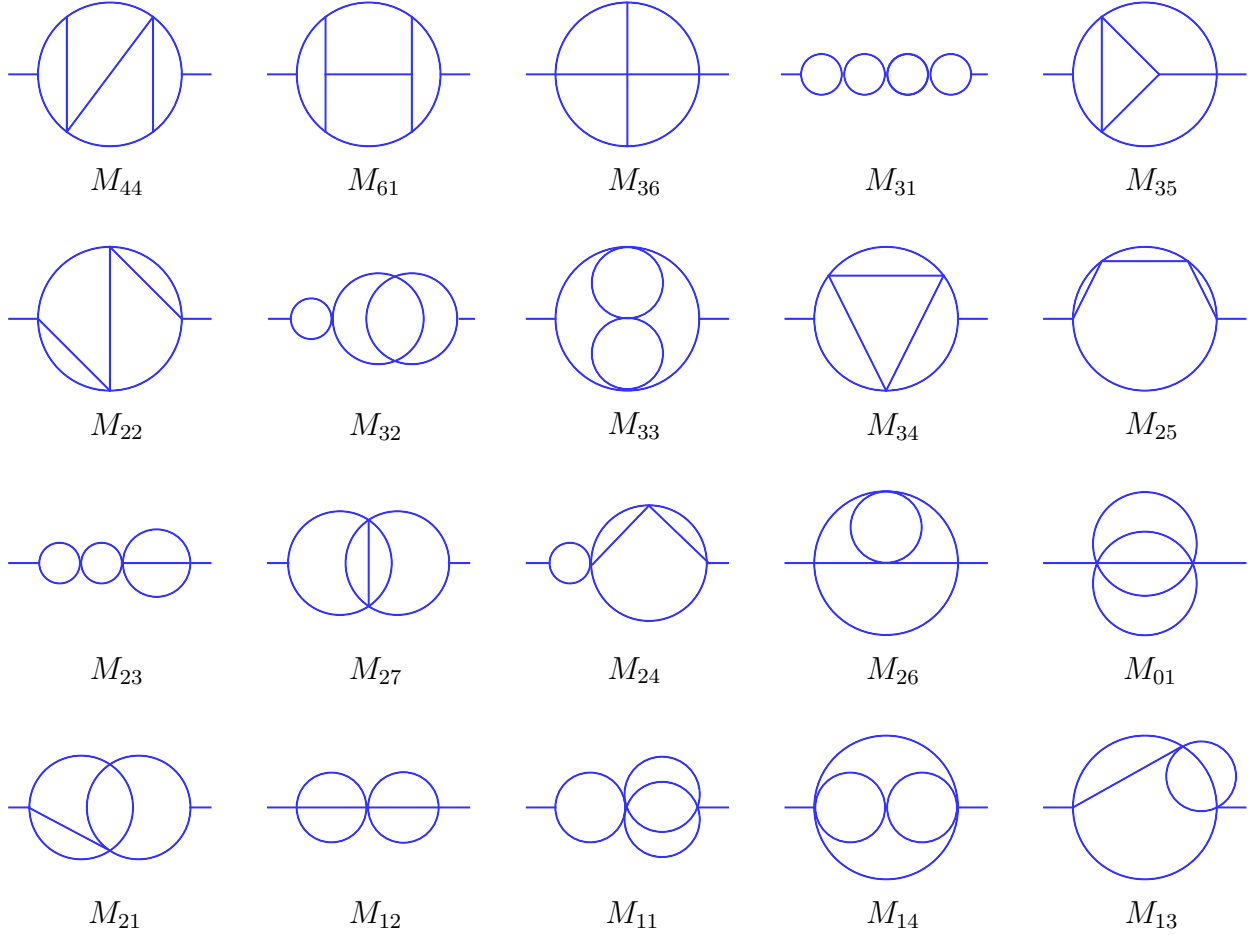


Figure 5: Diagrammatic representation of the planar basis integrals in the dual momentum representation. Blue line denote momentum propagators $1/k^2$ with the momentum $k = x_i - x_j$.

Finally, we combine together the relations (5.9) – (5.11), make use of the results of Ref. [27] for the master integrals shown in Fig. 5 and obtain from (5.8) the following result for C_4 or equivalently, five-loop Konishi anomalous dimension

$$\gamma_{\mathcal{K}}^{(5)} = -C_4 = \frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}\zeta_3^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7. \quad (5.12)$$

This relation is the main result of the paper. It is in perfect agreement with the prediction of the integrable models [5, 6, 7, 8].

6 Conclusions

In this paper we have developed a new efficient method for the computation of the Konishi anomalous dimension at higher loops. It does not use the conventional Feynman diagram technique with the associated very large number of contributing graphs and Feynman integrals. Instead, we exploited the recently discovered new symmetry of the four-point correlation function of $\mathcal{N} = 4$ SYM stress-tensor multiplets to predict the form of its integrand as a linear combination of a small number of relevant diagrams. Then, we examined the asymptotic behaviour of the logarithm of the four-point correlation function in the double short-distance limit and related the

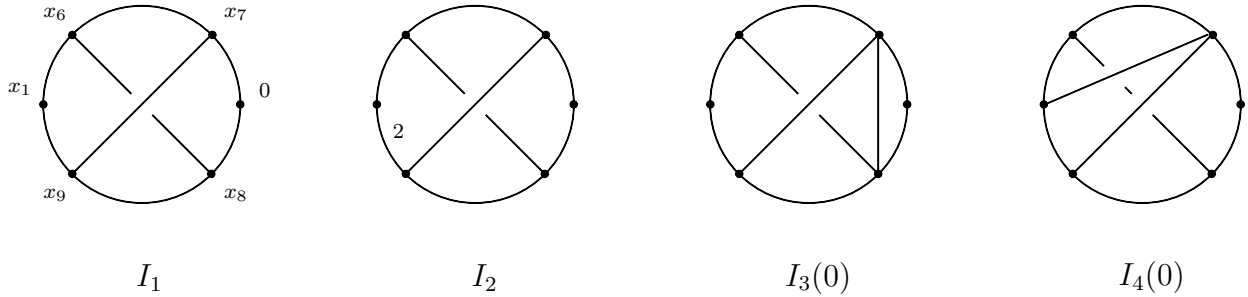


Figure 6: Diagrammatic representation of the non-planar basis integrals I_1 , I_2 and two auxiliary integrals $I_3(0)$, $I_4(0)$ defined in Eqs. (B.1) and (B.10), respectively. The line with the index 2 denotes a square of scalar propagator $1/(x^2)^2$, while all the remaining lines stand for $1/x^2$. The points x_1 and 0 are external and integration goes over the points x_6, \dots, x_9 .

Konishi anomalous dimension to its leading logarithmic singularity. Finally, by analyzing the expected singularity of the logarithm of the correlation function in this limit, we were able to lower the loop order of the contributing Feynman integrals by one, that is to express the Konishi anomalous dimension at ℓ loops in terms of finite two-point integrals at $(\ell - 1)$ loops. Going through these steps, we obtained the five-loop Konishi anomalous dimension in the planar limit as a sum of 22 master four-loop two-point integrals. Replacing the master integrals by their explicit expressions we arrived at an analytic result for this anomalous dimension which agrees with the integrability prediction [5, 6, 7, 8].

At present, the expression for the integrand of the four-point correlation function is known up to six loops in the planar limit [2]. In our calculation of the Konishi anomalous dimension we made use of this expression to five loops only. By applying the method developed in this paper, it is straightforward to extend the analysis to six loops and to express the six-loop Konishi anomalous dimension in terms of five-loop two-point integrals. The evaluation of such integrals is still an open problem, not because of their number, but because the IBP reduction to master integrals is a very complicated problem at this level. Still, we are optimistic that further development in this direction will eventually make the six-loop calculation possible. Likewise, no prediction for the six-loop Konishi anomalous dimension is available from AdS/CFT considerations, and to obtain it using the existing integrability approaches appears to be a rather non-trivial task.

Another result of our study is the prediction of the non-planar correction to the Konishi anomalous dimension at four loops in the form $r\zeta_5/N_c^2$ with r being an undetermined rational number. Our prediction is in full agreement with the result of the direct Feynman diagram calculation in [26], which also allows us to fix the value of $r = -135/2$. It is interesting to note that in our approach the non-planar correction at four loops originates from just a single and very simple three-loop propagator integral shown in Fig. 3. We would like to emphasize that the non-planar $O(1/N_c^2)$ correction to the four-point correlation function derived in Ref. [2] depends on four arbitrary rational constants. The parameter r is given by a particular linear combination of these coefficients. To fix each of the four coefficients we need three more relations. They can be obtained from the comparison of the non-planar corrections to the twist-two anomalous dimensions computed in Ref. [26] with the analogous results from the OPE analysis of the non-planar correction to the four-point correlation function. As explained in Ref. [2], the perturbative corrections to the correlation function have an iterative structure at higher loops. In application to the Konishi operator, this implies that the non-planar $O(1/N_c^2)$ correction to its anomalous

dimension is uniquely defined at all loops by the values of these four coefficients. We would like to mention that starting from five loops, the anomalous dimension receives $O(1/N_c^4)$ corrections. The method proposed in this paper can be equally applied to the study of such corrections.

The Konishi operator is just the first in an infinite series of twist-two operators, all appearing in the OPE of two $\mathcal{N} = 4$ SYM stress-tensor multiplets. The four-point function that we use for the evaluation of γ_K contains the information about the whole spectrum of anomalous dimensions of twist-two (as well as higher twist) operators. However, in order to extract it from the OPE, one needs to either evaluate analytically all relevant higher-loop four-point conformal integrals, or at least to work out their asymptotic expansion in the double short-distance limit. This problem is not yet solved in full generality beyond two loops¹², but it undoubtedly deserves further attention.

Acknowledgements

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Appendix A IR rearrangement in coordinate space

In this appendix we explain in detail the method that we employ in our calculation of the Konishi anomalous dimension. It represents an extension of the so-called infrared rearrangement method (IRR) [38] to the coordinate space.

To describe the method, let consider as an example the following four-loop integral in Euclidean D -dimensional space-time (with $D = 4 - 2\epsilon$)

$$I(x_{13}) = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{(x_{13}^2)^4 d^D x_5 \dots d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}. \quad (\text{A.1})$$

We would like to stress that x_i are true coordinates in the configuration space and, therefore, $I(x_{13})$ is different from the conventional integrals that one encounters in the dimensional regularization in which case all distances in the denominator appear in the power $(1 - \epsilon)$.

The integral (A.1) has a simple pole in ϵ

$$I(x_{13}) = (x_{13}^2)^{-4\epsilon} \left[\frac{C}{\epsilon} + O(\epsilon^0) \right]. \quad (\text{A.2})$$

It comes from the integration over the region where x_5, \dots, x_8 are all close to x_1 and from the symmetrical region where x_5, \dots, x_8 are all close to x_3 . Since the integration variables as true

¹²Although at three loops, the inverse process has been partially done, namely a prediction has been made for the three-loop correlation function in the limit $u \rightarrow 0$ but with finite v [37], by making use of the three-loop twist-two, arbitrary spin anomalous dimensions predicted in [24] together with known lower-loop twist-two data, and a conformal partial wave analysis. See appendix C for more on higher spin twist-two anomalous dimensions and the four-point correlation function.

coordinates in the Euclidean space, the pole $1/\epsilon$ has to be qualified as an UV divergence. Notice that the integrand of $I(x_{13})$ coincides (up to an overall normalization factor) with (4.27). As a result, the residue C defines the four-loop non-planar correction to the Konishi anomalous dimension.

In general, the UV divergences in the coordinate space come from regions where the integrand considered as a generalized function of x_i (tempered distribution, i.e. linear functional on a space of test functions) is ill-defined. In our example, the product of x^2 -factors in the denominator of (A.1) turns out to be unintegrable in a vicinity of the two external points, x_1 and x_3 . In the first case, we consider the product

$$F(x_1, x_5, \dots, x_8) = \frac{1}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} \quad (\text{A.3})$$

as a tempered distribution. Its divergent part is described by an UV counter-term¹³

$$\Delta(x_1, x_5, \dots, x_8) = \frac{C}{2\epsilon} \delta(x_1 - x_5) \dots \delta(x_1 - x_8), \quad (\text{A.4})$$

with the constant C determined below. Similar counter-term $\Delta(x_3, x_5, \dots, x_8)$ describes singular behaviour of the integrand (A.1) in the vicinity of x_3 . Thus, the pole part of (A.1) is just twice the factor $C/(2\epsilon)$ in (A.4)

$$I = \int d^D x_5 \dots d^D x_8 [\Delta(x_1, x_5, \dots, x_8) + \Delta(x_3, x_5, \dots, x_8)] + O(\epsilon^0) = \frac{C}{\epsilon} + O(\epsilon^0), \quad (\text{A.5})$$

leading to (A.2).

To evaluate the constant C in (A.4) we apply the infrared rearrangement (IRR) method originally proposed by Vladimirov in Ref. [38] in the momentum space. It makes use of the fact that, for an infrared finite but logarithmically UV-divergent Feynman integral without subdivergences, the contribution of the counter-term is just a constant. The idea of IRR is to set the external momenta to zero and then, in order to avoid the appearance of IR divergences, to introduce an external momentum (or a mass) in such a way that the calculation becomes simpler.¹⁴

Applying the IRR method to (A.1), we should have transformed the integral $I(x_{13})$ to the momentum space via Fourier transform. However we will not do this for the following two reasons. First, the resulting momentum integral will be four-loop one while we can obtain the residue C from three-fold integral only as described below. It is well known [20] that the evaluation of the UV pole part of a given ℓ -loop momentum-space Feynman integral can be reduced to evaluating massless propagator $(\ell - 1)$ -loop Feynman integrals to order ϵ^0 . However, as was already mentioned, the integral (A.1) is different from the conventional Feynman integral. In particular, the $1/x^2$ factors on the right-hand side of (A.1) are replaced in the momentum representation by the factors of $1/(k^2)^{1-\epsilon}$ depending on ϵ . These are much more complicated objects, both from the point of view of an IBP reduction and evaluating master integrals, so that it is the second reason why we want to stay in coordinate space.

Let us apply the IRR method to (A.3) in the coordinate space and treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points. Notice that setting an external momentum to zero

¹³In this example, Δ contains a simple pole in ϵ . In a general situation, this would be a finite linear combination of negative powers of ϵ .

¹⁴If it is not possible to avoid such IR divergences one can remove them immediately by the so-called R^* -operation [39, 20] but we do not meet such a complication in our calculations.

corresponds to integrating over the corresponding coordinate. Then, the constant C in (A.4) can be obtained by integrating the both sides of (A.3) with respect to internal points

$$F(x_1, x_5) = \int \frac{d^D x_5 d^D x_6 d^D x_7}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} = \frac{C}{2\epsilon} \delta(x_1 - x_5) + O(\epsilon^0). \quad (\text{A.6})$$

The integral on the left-hand side depends on the two external points and is of propagator type. We can check it has no IR divergences, i.e. divergences at large values of coordinates, and has the following form by dimensional arguments

$$F(x_1, x_5) = f(\epsilon) \frac{1}{(x_{15}^2)^{2+3\epsilon}}. \quad (\text{A.7})$$

Here the only source of the simple pole in ϵ is hidden in the second factor (which is considered as a distribution) so that $f(\epsilon)$ is analytic in a vicinity of the point $\epsilon = 0$. The simplest way to reveal the $1/\epsilon$ pole of the distribution $1/(x_{15}^2)^{2+3\epsilon}$ is to take its D -dimensional Fourier transform with a help of the identity

$$\mathcal{F} \left[\frac{1}{(x^2)^\lambda} \right] = \frac{1}{\pi^{D/2}} \int d^D x \, e^{ipx} \frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda} \Gamma(D/2 - \lambda)}{\Gamma(\lambda)} \frac{1}{(p^2)^{D/2-\lambda}}. \quad (\text{A.8})$$

In particular, for $\lambda = 2 + 3\epsilon$ we find from (A.7) (for $x_5 = 0$)

$$\mathcal{F} [F(x_1, 0)] = f(\epsilon) \frac{4^{-4\epsilon} \Gamma(-4\epsilon)}{\Gamma(2 + 3\epsilon)} \frac{1}{(p^2)^{-4\epsilon}} = -\frac{f(0)}{4\epsilon} + O(\epsilon^0). \quad (\text{A.9})$$

At the same time, replacing $F(x_1, 0)$ by its expression (A.6) we obtain the left-hand side of this relation as $C/(2\epsilon) + O(\epsilon^0)$ leading to

$$C = -\frac{1}{2} f(0) = -\frac{1}{2} F(x_1, 0) \Big|_{x_1^2=1, D=4}. \quad (\text{A.10})$$

It is easy to see that the integral $F(x_1, x_5)$, Eq. (A.6), corresponds to a planar graph shown in Fig. 3. After going to the dual momenta, we find that it coincides with the graph N_2 of Baikov and Chetyrkin [27] leading to

$$C = -10 \zeta(5). \quad (\text{A.11})$$

Appendix B Non-planar master integrals

In this appendix, we evaluate the two non-planar master Euclidean integrals (5.10). They have the following form (with $D = 4 - 2\epsilon$ and $x_1^2 = 1$)

$$\begin{aligned} I_1 &= \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_1}{\epsilon} + b_1 + c_1 \epsilon + O(\epsilon^2), \\ I_2 &= \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2 \epsilon + O(\epsilon^2). \end{aligned} \quad (\text{B.1})$$

Here we introduced the factor in front of the integrals to avoid the appearance of terms proportional to $\ln \pi$ and Euler's constant γ in the right-hand side. The diagrammatic representation of I_1 and I_2 is shown in Fig. 6. Both integrals develop poles $1/\epsilon$ but their origin is different. For the integral I_1 it comes from integration over x_6, x_7, x_8, x_9 going to infinity simultaneously and, therefore, has an IR origin. For the integral I_2 the pole comes from integration at short distances $x_{19} \rightarrow 0$ and has UV origin.¹⁵

Substituting Eqs.(5.9) – (5.11) and (B.1) into (5.8) and making use of the results of Ref. [27] we finally obtain the following expression for C_4

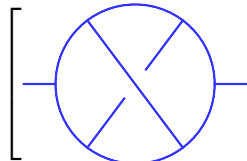
$$\begin{aligned}
C_4 = & \left(\frac{3a_1}{80} + \frac{9a_2}{160} + \frac{15\zeta_5}{16} \right) \epsilon^{-2} \\
& + \left(-\frac{21a_1}{80} - \frac{9a_2}{80} + \frac{3b_1}{80} + \frac{9b_2}{160} + \frac{15\zeta_3^2}{16} + \frac{5\pi^6}{2016} \right) \epsilon^{-1} \\
& + \left(\frac{741a_1}{640} + \frac{807a_2}{320} - \frac{21b_1}{80} - \frac{9b_2}{80} + \frac{3c_1}{80} + \frac{9c_2}{160} - \frac{225\zeta_7}{64} - \frac{5\pi^2\zeta_5}{16} \right. \\
& \quad \left. + \frac{7035\zeta_5}{128} + \frac{81\zeta_3^2}{16} + \frac{\pi^4\zeta_3}{32} - \frac{27\zeta_3}{4} - \frac{237}{16} \right) + O(\epsilon). \quad (B.2)
\end{aligned}$$

Here the constants a_i, b_i and c_i describe the contribution of the two non-planar master integrals, Eq. (B.1). We recall that C_4 defines the five-loop correction to the Konishi anomalous dimension and, therefore, it should be finite for $\epsilon \rightarrow 0$. The condition for the $1/\epsilon^2$ and $1/\epsilon$ poles to cancel inside C_4 leads to two relations between the coefficients a_i and b_i . As we shall see in a moment, these relations are indeed satisfied.

Let us start with the leading $O(1/\epsilon)$ term on the right-hand side of (B.1). The simplest way to compute the residue at the pole is to Fourier transform the integral into momentum with the help of (A.8). Notice that the expressions on the right-hand side of (B.1) are valid for $x_1^2 = 1$, but their dependence on x_1^2 can easily be restored from dimension analysis. In this way, we find from the first relation in (B.1)

$$\mathcal{F}[I_1] = \mathcal{F} \left[\frac{a_1}{\epsilon} (x_1^2)^{-4\epsilon} + O(\epsilon^0) \right] = (64 a_1 + O(\epsilon)) (p^2)^{-2+5\epsilon}. \quad (B.3)$$

This relation implies that the coefficient a_1 can be obtained from the Fourier transformed integral $\mathcal{F}[I_1]$ evaluated at $D = 4$ dimensions. Transforming the integral I_1 into the momentum representation we find that $\mathcal{F}[I_1]$ coincides (up to a factor of 16) with the conventional four-dimensional momentum Feynman integral denoted N_0 in Ref. [27]

$$\mathcal{F}[I_1] = 16 \left[\text{Diagram} \right] = 16 \times (20\zeta_5 + O(\epsilon)) (p^2)^{-2+5\epsilon}. \quad (B.4)$$


Comparing this relation with (B.3) we find

$$a_1 = 5\zeta_5. \quad (B.5)$$

¹⁵Since the Euclidean integrals in (B.1) are positive definite, this explain why their residues at the pole, a_1 and a_2 , have opposite signs (see Eqs. (B.5) and (B.9) below).

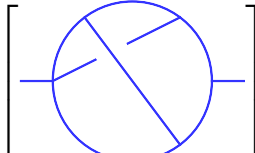
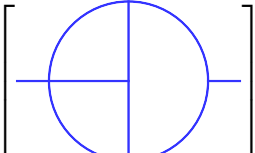
Let us now turn to the integral I_2 in (B.1) and Fourier transform it

$$\mathcal{F}[I_2] = F \left[\frac{a_2}{\epsilon} (x_1^2)^{-1-4\epsilon} + O(\epsilon^0) \right] = 4 \left(\frac{a_2}{\epsilon} + O(\epsilon) \right) (p^2)^{-1+5\epsilon}. \quad (\text{B.6})$$

To identify the momentum integral corresponding to $\mathcal{F}[I_2]$ we have to Fourier transform all factors in the denominator of I_2 including $1/(x_{19}^2)^2$. In that case, we find from (A.8)

$$\mathcal{F} \left[\frac{1}{(x_{19}^2)^2} \right] = 2^{-2\epsilon} \Gamma(-\epsilon) (p^2)^\epsilon = -\frac{1}{\epsilon} + O(\epsilon^0). \quad (\text{B.7})$$

The fact that the residue at the pole in this relation does not depend on the momentum p implies that the corresponding line in the Feynman diagram shrinks to a point. As a result,

$$\mathcal{F}[I_2] = -\frac{4}{\epsilon} \left[\text{Diagram 1} \right] = -\frac{4}{\epsilon} \left[\text{Diagram 2} \right] = -\frac{4}{\epsilon} \times (20\zeta_5 + O(\epsilon)) (p^2)^{-1+5\epsilon}. \quad (\text{B.8})$$



Here in the second relation we redrew the same diagram, so that it takes the form of the diagram N_2 in the notation of Ref. [27]. Comparing the last relation with (B.6) we conclude that

$$a_2 = -20\zeta_5. \quad (\text{B.9})$$

Let us now compute the subleading terms in the expansion (B.1). To this end, we introduce the following auxiliary integrals (with $D = 4 - 2\epsilon$)

$$\begin{aligned} I_3(\kappa) &= \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}}, \\ I_4(\kappa) &= \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{17}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}}, \end{aligned} \quad (\text{B.10})$$

with κ being a parameter. The diagrammatic representation for these integrals (for $\kappa = 0$) is shown in Fig. 6. For finite κ the two integrals are finite as $\epsilon \rightarrow 0$ and, therefore, they admit an expansion in powers of ϵ and $\kappa\epsilon$

$$I_i(\kappa) = b_i + \epsilon(c_i + \kappa d_i) + O(\epsilon^2), \quad (i = 3, 4), \quad (\text{B.11})$$

where we put $x_1^2 = 1$. Notice that the leading $O(\epsilon^0)$ term does not depend on κ , whereas the $O(\epsilon)$ term is a linear of function of κ .

A distinguishing feature of the integrals (B.10) is that, in the special case $\kappa = 0$, the IBP relations allow us express $I_3(0)$ and $I_4(0)$ in terms of two master integrals I_1 and I_2 , Eqs. (B.1). More precisely,

$$\begin{aligned} b_3 &= -\frac{2}{3} b_1 - \frac{7}{3} b_2 - 70\zeta_5 + \frac{26}{3}\zeta_3^2 - \frac{65}{567}\pi^6, \\ b_4 &= -b_1 - 2b_2 - 45\zeta_5 + 7\zeta_3^2 - \frac{5}{54}\pi^6, \\ c_3 &= \frac{14}{3} b_1 + \frac{14}{3} b_2 - \frac{2}{3} c_1 - \frac{7}{3} c_2 - \frac{4667}{6}\zeta_7 + \frac{130}{9}\pi^2\zeta_5 - \frac{100}{3}\zeta_5 + \frac{13}{45}\pi^4\zeta_3, \\ c_4 &= 2b_1 - 6b_2 - c_1 - 2c_2 - \frac{4193}{4}\zeta_7 + \frac{35}{3}\pi^2\zeta_5 - 275\zeta_5 + 35\zeta_3^2 + \frac{7}{30}\pi^4\zeta_3 - \frac{25}{54}\pi^6. \end{aligned} \quad (\text{B.12})$$

Once the auxiliary integrals (B.11) have been computed, we could use these relations to obtain the needed coefficients b_1, b_2 and c_1, c_2 .

One may wonder why we introduced the parameter κ into the definition of the integrals (B.10) if we only need its value at $\kappa = 0$. The reason for this is that, as we will see in a moment, it is much easier to compute the integrals (B.10) for the two special values $\kappa = 1/2$ and $\kappa = 1$. Then, taking into account that $I_i(\kappa)$ is a linear function of κ at order $O(\epsilon)$, Eq. (B.11), we find

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2). \quad (\text{B.13})$$

In what follows, we shall evaluate $I_i(1)$ and $I_i(1/2)$ and, then, apply this relation to compute b_3, b_4 and c_3, c_4 .

Let us consider the integrals (B.10) for $\kappa = 1$. In this case it is easy to see that the integrand is given by a product of factors $1/(x^2)^{1-\epsilon}$ which coincide with scalar propagators in $D = 4 - 2\epsilon$ dimensions. As a result, upon the Fourier transform, the integrals $\mathcal{F}[I_3(1)]$ and $\mathcal{F}[I_4(1)]$ are given by conventional four-loop momentum Feynman integrals. In this way, we find that the integral $\mathcal{F}[I_3(1)]$ coincides with the master integral M_{45} in the notation of [27]

$$I_3(1) = G_0^4 M_{45} = 36\zeta_3^2 + \epsilon(108\zeta_3\zeta_4 + 288\zeta_3^2 - 378\zeta_7) + O(\epsilon^2), \quad (\text{B.14})$$

where the additional factor $G_0 = e^{\gamma\epsilon} \Gamma(1+\epsilon)\Gamma(1-\epsilon)^2/\Gamma(2-2\epsilon) = 1 + 2\epsilon + O(\epsilon^2)$ is inserted to convert the result of Ref. [27] obtained in the G -scheme to the regularization scheme used in (B.10). The second momentum integral $\mathcal{F}[I_4(1)]$ is not a master integral. We applied FIRE to reduce it to the master integrals of Ref. [27] and arrived at the following result:

$$\begin{aligned} I_4(1) = & -M_{01} \frac{(3-4\epsilon)(1-4\epsilon)(4-5\epsilon)(3-5\epsilon)(10-105\epsilon+326\epsilon^2-319\epsilon^3)}{6(1-\epsilon)\epsilon^5(1-3\epsilon)} \\ & + M_{11} \frac{4(1-2\epsilon)(2-3\epsilon)(1-3\epsilon)(3-4\epsilon)(1-4\epsilon)}{3(1-\epsilon)\epsilon^4} - M_{35} \frac{2(1-4\epsilon)(1-5\epsilon)}{3(1-\epsilon)\epsilon} \\ & - M_{13} \frac{(1-2\epsilon)(2-3\epsilon)(3-5\epsilon)(2-9\epsilon)(7-19\epsilon)}{6(1-\epsilon)\epsilon^4} - M_{36} \frac{1-5\epsilon}{1-\epsilon} \\ & + M_{12} \frac{(1-2\epsilon)(2-3\epsilon)^2(1-3\epsilon)^2}{(1-\epsilon)\epsilon^4} + M_{21} \frac{4(1-2\epsilon)^3(1-4\epsilon)}{3(1-\epsilon)\epsilon^3}. \end{aligned} \quad (\text{B.15})$$

Replacing the basis integrals by their explicit expressions we get

$$I_4(1) = 36\zeta_3^2 + \epsilon \left(108\zeta_3\zeta_4 + 108\zeta_3^2 + \frac{189}{2}\zeta_7 \right) + O(\epsilon^2), \quad (\text{B.16})$$

where we put $x_1^2 = 1$.

Let us now examine the integrals (B.10) for $\kappa = 1/2$. In this case, the special feature of the integral $I_3(1/2)$ is that the conformal weight of the integrand at the integration points x_7 and x_8 equals the space-time dimension $4(1-\kappa\epsilon) = D$. As a consequence, performing inversion $x_i^\mu \rightarrow x_i^\mu/x_i^2$ we obtain the following representation for $I_3(1/2)$ (at $x_1^2 = 1$)

$$I_3(1/2) = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{66}^2 x_{89}^2)^{1-\epsilon/2}} = \text{Diagram} \quad , \quad (\text{B.17})$$

where on the right-hand side we depicted the corresponding Feynman diagram. Compared with the first relation in (B.10), the product $x_7^2 x_8^2$ gets replaced here with $x_6^2 x_9^2$. We observe that the diagram on the right-hand side of (B.17) contains a two-loop subgraph. As a consequence, the integration over x_7 and x_8 can be easily performed with the help of identity [31]

$$\frac{e^{2\gamma\epsilon}}{\pi^D} \int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{1}{(x_{69}^2)^{1-\epsilon/2}} [6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)]. \quad (\text{B.18})$$

Substituting this relation into (B.17) we find that the remaining integral over x_6 and x_9 takes the same form as (B.18) leading to

$$I_3(1/2) = [6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)]^2. \quad (\text{B.19})$$

It remains to determine $I_4(1/2)$. The corresponding Feynman diagram has the same form as the one for $I_4(1/2)$ (see Fig. 6) with the only difference that each solid line carries the index $(1-\epsilon/2)$. It is easy to see that the integrals $I_4(1/2)$ and $I_3(1/2)$ look differently. Quite remarkably, as we will show later in this appendix, they coincide leading to

$$I_4(1/2) = I_3(1/2) = 36\zeta_3^2 + \epsilon(108\zeta_3\zeta_4 + 144\zeta_3^2) + O(\epsilon^2). \quad (\text{B.20})$$

We would like to stress that this relation is exact and it holds for arbitrary ϵ .

Then, we combine the relations (B.14), (B.16), (B.20) together and find from (B.11)

$$\begin{aligned} I_3(\kappa) &= 36\zeta_3^2 + \epsilon(108\zeta_3\zeta_4 + 288\kappa\zeta_3^2 + (1-2\kappa)378\zeta_7) + O(\epsilon^2), \\ I_4(\kappa) &= 36\zeta_3^2 + \epsilon(108\zeta_3\zeta_4 + (180-72\kappa)\zeta_3^2 - \frac{189}{2}(1-2\kappa)\zeta_7) + O(\epsilon^2). \end{aligned} \quad (\text{B.21})$$

Matching these expressions into (B.11) we obtain the following relations for the coefficients

$$\begin{aligned} b_3 &= b_4 = 36\zeta_3^2, \\ c_3 &= 108\zeta_3\zeta_4 + 378\zeta_7, \\ c_4 &= 108\zeta_3\zeta_4 + 180\zeta_3^2 - \frac{189}{2}\zeta_7. \end{aligned} \quad (\text{B.22})$$

Their substitution into (B.12) yields a system of linear relations for the coefficients b_1, b_2 and c_1, c_2 whose solution is

$$\begin{aligned} a_1 &= 5\zeta_5, & b_1 &= \frac{5}{378}\pi^6 - 13\zeta_3^2 + 35\zeta_5, \\ a_2 &= -20\zeta_5, & b_2 &= -\frac{10}{189}\pi^6 - 8\zeta_3^2 - 40\zeta_5, \\ c_1 &= -\frac{13}{30}\pi^4\zeta_3 - 91\zeta_3^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6, \\ c_2 &= -\frac{4}{15}\pi^4\zeta_3 - 16\zeta_3^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6. \end{aligned} \quad (\text{B.23})$$

Substituting these relations in (B.2) we verify the cancellation of poles in ϵ and reproduce (5.12).

We complete this appendix with a proof of the relation $I_4(1/2) = I_3(1/2)$. It relies on applying the cut-and-glue method of Ref. [33, 27]¹⁶. Let us examine the integrals (B.10) for $\kappa = 1/2 + \lambda/(10\epsilon)$ with λ arbitrary. Dimensional analysis shows that the integrals have the following form

$$I_i(1/2 + \lambda/(10\epsilon)) = \frac{c_i(\epsilon, \lambda)}{(x_1^2)^{1-\epsilon/2-9\lambda/10}}, \quad (i = 3, 4), \quad (\text{B.24})$$

with c_i being some function of ϵ and λ . Then, the relation $I_4(1/2) = I_3(1/2)$ implies that for arbitrary ϵ

$$c_3(\epsilon, 0) = c_4(\epsilon, 0). \quad (\text{B.25})$$

To show this, we consider the following Fourier integral

$$\mathcal{F} \left[\frac{I_i(1/2 + \lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}} \right] = \mathcal{F} \left[\frac{c_i(\epsilon, \lambda)}{(x_1^2)^{2-\epsilon-\lambda}} \right] = c_i(\epsilon, \lambda) \frac{2^{-2\lambda}\Gamma(\lambda)}{\Gamma(2-\epsilon-\lambda)} (p^2)^{-\lambda}, \quad (\text{B.26})$$

where in the second relation we applied (A.8). A crucial observation is that for small λ the expression on the right-hand side develops a pole $1/\lambda$ with the residue independent on the momentum p

$$\mathcal{F} \left[\frac{I_i(1/2 + \lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}} \right] = \lambda^{-1} \frac{c_i(\epsilon, 0)}{\Gamma(2-\epsilon)} + O(\lambda^0). \quad (\text{B.27})$$

The integrals $I_i(1/2 + \lambda/(10\epsilon))$ (with $i = 3, 4$) are described by the Feynman diagrams shown in Fig. 7 in the left column. All solid lines in these diagrams correspond to factors of $1/(x^2)^{1-\epsilon/2-\lambda/10}$. Then, dividing I_i by $(x_1^2)^{1-\epsilon/2-\lambda/10}$ amounts to adding one additional line to the diagram connecting the external points x_1 and 0. The resulting diagrams are shown in the right column of Fig. 7. Notice that, up to changing the labels of the dots, these two graphs coincide (to see this, it suffices to rotate the lower diagram clock-wise by $2\pi/3$). After taking the Fourier transform, in the momentum representation, the momenta p and $-p$ are injected into the external points x_1 and 0, respectively, while for the remaining integrated points the corresponding momentum equals zero. As was argued in [27], the very fact that the leading asymptotic behaviour of the Fourier integral for $\lambda \rightarrow 0$ does not depend on the momentum p implies that the residue at the pole $1/\lambda$ is not sensitive to the choice of the external points. This is exactly what happens for the two graphs in the right column of Fig. 7, the only difference between these graphs is in the assignment of the external points. Since their residue at the pole $1/\lambda$ are defined by the functions $c_3(\epsilon, 0)$ and $c_4(\epsilon, 0)$, we conclude that they are equal to each other leading to (B.25).

To check numerically our analytic results for these two non-planar integrals we used the code FIESTA [41] which gave the precision of six digits.

Appendix C Twist-two anomalous dimensions

In Section 2.2, we have applied the OPE (2.8) to identify the contribution of the Konishi operator to the four-point correlation function in the double-short distance $x_2 \rightarrow x_1$ and $x_4 \rightarrow x_3$. In this

¹⁶It turns out that this generalized gluing is very close in its spirit to the strategy of ref. [40] where Feynman integrals were considered as distributions with respect to the parameter of analytic regularization.

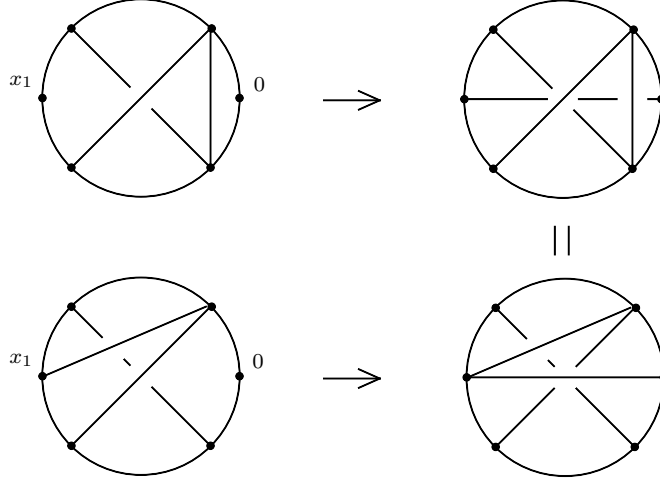


Figure 7: Glue procedure for the integrals I_3 and I_4 defined in (B.24). All lines correspond to factors of $1/(x^2)^\alpha$ with the same index $\alpha = 1 - \epsilon/2 - \lambda/10$. The dots with the labels x_1 and 0 describe the external points, the remaining four dots describe the integration points.

appendix, we address the larger class of operators of twist two, of which the Konishi operator is the simplest (spin zero) representative.

For the *protected* scalar operators (2.2) the OPE takes the following general form:

$$\mathcal{O}(x_1, y_1)\mathcal{O}(x_2, y_2) = \sum_{\Delta, S, \mathcal{R}} \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_\Delta}^{\mathcal{R}}(y_1, y_2)}{(x_{12}^2)^{2-\frac{1}{2}(\Delta-S)}} (x_{12})_{\mu_1} \dots (x_{12})_{\mu_S} \left[O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}(x_2) + \dots \right]. \quad (\text{C.1})$$

Here the sum on the right-hand side runs over conformal primary operators $O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}$ carrying Lorentz spin S , scaling dimension Δ and the dots denote the contribution of their conformal descendants. The relation (C.1) generalizes (2.8) which describes the most singular contribution of operators with the lowest value of Δ . Also, since each operator $\mathcal{O}(x_i, y_i)$ belongs to the representation $\mathbf{20}'$ of the R symmetry group $SU(4)$, the right-hand side of (C.1) involves the sum over all irreducible representations \mathcal{R} that appear in the tensor product

$$\mathbf{20}' \times \mathbf{20}' = \mathbf{1} + \mathbf{15} + \mathbf{20}' + \mathbf{84} + \mathbf{105} + \mathbf{175}. \quad (\text{C.2})$$

These representations can be identified from the y -dependence of the operators $O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}(x_2)$ in the expansion. The contribution of each operator to the right-hand side of (C.1) is accompanied by the coefficient function $C_{\mathcal{O}\mathcal{O}\mathcal{O}_\Delta}^{\mathcal{R}}$. It determines the three-point correlation function $\langle \mathcal{O}(1)\mathcal{O}(2)O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}(x_3) \rangle$ and depends, in general, on the coupling constant.

A notable example of the operators $O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}$ that shall play a special role in our discussion are the twist-two operator \mathcal{O}_S . They are $SU(4)$ singlet bilinear operators with arbitrary (even) Lorentz spin S and naive scaling dimension $\Delta^{(0)} = 2 + S$. The Konishi operator is the special case of such operators with spin zero, $S = 0$. The scaling dimensions of the twist-two operators acquire an anomalous contribution:

$$\Delta_S = S + 2 + \gamma_S(a) = S + 2 + \sum_{\ell=1}^{\infty} a^\ell \gamma_S^{(\ell)}, \quad (\text{C.3})$$

with $\gamma_S^{(\ell)}$ being non-trivial functions of S .

Applying (C.1), we find that every operator contributing to the OPE gives a definite contribution to the four-point function (2.3) known as a conformal partial wave amplitude (or CPWA) [42]

$$G(1, 2, 3, 4) = \frac{1}{x_{12}^4 x_{34}^4} \sum_{\Delta, S, \mathcal{R}} G_{\Delta, S}^{(\mathcal{R})}(u, v), \quad (\text{C.4})$$

where $G_{\Delta, S}^{(\mathcal{R})}(u, v)$ describes the contribution of the conformal primary operator $O_{\Delta}^{\mu_1 \dots \mu_S; \mathcal{R}}$ and its conformal descendants. The conformal partial waves $G_{\Delta, S}^{(\mathcal{R})}(u, v)$ are definite functions of the conformal ratios u and v defined in (2.13). For $u \rightarrow 0$ and $v \rightarrow 1$ they have the following asymptotic behaviour [43]

$$G_{\Delta, S}^{(\mathcal{R})}(u, v) \sim u^{(\Delta-S)/2} (1-v)^S [1 + O(u, 1-v)]. \quad (\text{C.5})$$

Expanding the known result for the four-point correlation function over the CPWA and applying (C.5), we can extract the anomalous dimension of the operator. For more information see, for example [44, 43].

In $\mathcal{N} = 4$ SYM, the twist-two operators are the ground states (or superconformal primaries) of “long” (or unprotected) supermultiplets. Each state (or superdescendant) in such a multiplet has different naive conformal dimension, Lorentz spin and $SU(4)$ quantum numbers, but they all share the same anomalous dimension $\gamma_S(a)$. Therefore, to determine the anomalous dimension of the twist-two operators, we may look at the state in the multiplet which is most convenient to identify in the CPWA expansion (C.4). As pointed out in [43], the best choice is the twist-six state corresponding to the $SU(4)$ channel **105** in (C.2). The advantage of this choice is that the twist-two supermultiplet has only one state in this $SU(4)$ channel, while for all other choices there are multiple candidates.¹⁷

The specific correlation function which singles out the state in the **105** has the form

$$G_4(1, 2, 3, 4) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \tilde{\mathcal{O}}(x_3) \tilde{\mathcal{O}}(x_4) \rangle, \quad (\text{C.6})$$

where $\mathcal{O} = \text{tr}(ZZ)$, $\tilde{\mathcal{O}} = \text{tr}(\bar{Z}\bar{Z})$ and $Z = \Phi^1 + i\Phi^2$ is a complex scalar field. It can be obtained from the general expression for the correlation function (2.3) by choosing the harmonic variables as $Y_1 = Y_2 = (1, i, 0, 0, 0, 0)$ and $Y_3 = Y_4 = (1, -i, 0, 0, 0, 0)$. Then, the relations (2.3) and (2.4) take the following form

$$G_4(1, 2, 3, 4) = \frac{2(N_c^2 - 1)}{(4\pi^2)^4} \times \frac{1}{x_{13}^2 x_{24}^2} \times \frac{u}{v} \times \sum_{\ell \geq 1} a^\ell F^{(\ell)}(x_i), \quad (\text{C.7})$$

where the additional factor of u/v comes from the function $R(1, 2, 3, 4)$ (see Ref. [1]). Then, the

¹⁷Choosing the state in the **105** is helpful, but not indispensable for carrying out the CPWA analysis of the twist-two operators. The two-loop anomalous dimension of the $SU(4)$ singlet state of spin two was found for the first time in [44], using a form of the CPWA expansion different from that in [43].

CPWA expansion (C.4) of the correlation function (C.6) is (for $u \rightarrow 0$)¹⁸

$$G_4(1, 2, 3, 4) \sim \frac{N_c^2 - 1}{(4\pi^2)^4} \frac{1}{(x_{12}^2 x_{34}^2)^2} \sum_{S/2 \in \mathbb{Z}^+} A_S(a) u^{3+\gamma_S(a)/2} \bar{v}^S \\ \times {}_2F_1\left(3 + S + \frac{1}{2}\gamma_S(a), 3 + S + \frac{1}{2}\gamma_S(a), 6 + 2S + \gamma_S(a) | \bar{v}\right) . \quad (\text{C.8})$$

where $\bar{v} = 1 - v$ and the sum runs over non-negative even spins S and the coefficient function $A_S(a)$ and the anomalous dimension $\gamma_S(a)$ are functions of the 't Hooft coupling a

$$A_S(a) = \sum_{\ell \geq 1} a^\ell A_S^{(\ell)} , \quad \gamma_S(a) = \sum_{\ell \geq 1} a^\ell \gamma_S^{(\ell)} . \quad (\text{C.9})$$

Comparing (C.7) and (C.8) we thus obtain (for $u \rightarrow 0$)

$$2x_{13}^2 x_{24}^2 \sum_{\ell \geq 1} a^\ell F^{(\ell)} = \sum_{S/2 \in \mathbb{Z}^+} A_S u^{\gamma_S/2} (1 - \bar{v}) \bar{v}^S {}_2F_1\left(3 + S + \frac{1}{2}\gamma_S, 3 + S + \frac{1}{2}\gamma_S, 6 + 2S + \gamma_S | \bar{v}\right) . \quad (\text{C.10})$$

To make use of this relation, we expand both sides of this relation in powers of the coupling constant a and compare the coefficients of the powers of $\ln u$. Furthermore, on the right-hand side, the contribution of the operator with spin S is suppressed by a factor of \bar{v}^S . Therefore, expanding both sides of (C.10) for $\bar{v} \rightarrow 0$ up to $O(\bar{v}^S)$ terms and equating the coefficients of the powers of $\ln u$ and \bar{v} , we obtain equations for the expansion coefficients of $A_{S'}(a)$ and $\gamma_{S'}(a)$ for $S' \leq S$.

The main goal of this appendix is to apply (C.10) to extract the coefficients (C.9) at three loops. To this end, we have to supplement (C.10) with the expressions for the correlation function $F^{(\ell)}$ for $\ell = 1, 2, 3$ in the OPE limit. At one and two loops, the corresponding expressions for $F^{(1)}$ and $F^{(2)}$, Eq. (3.1), are known analytically [30]. At three loops, we can use the result for $F^{(3)}$ recently found in Ref. [1] (and described in section 4.2). The explicit expression for $F^{(3)}$ in terms of the basis scalar integrals is

$$F^{(3)} = 2g(1, 2, 3, 4) [x_{12}^2 x_{34}^2 h(1, 2; 3, 4) + x_{13}^2 x_{24}^2 h(1, 3; 2, 4) + x_{14}^2 x_{23}^2 h(1, 4; 2, 3)] \\ + 6[L(1, 2; 3, 4) + L(1, 3; 2, 4) + L(1, 4; 2, 3)] \\ + 4[E(1; 3, 4; 2) + E(1; 2, 4; 3) + E(1; 2, 3; 4)] \\ + (1 + 1/v) H(1, 2; 3, 4) + (1 + u/v) H(1, 3; 2, 4) + (1 + u) H(1, 4; 2, 3) . \quad (\text{C.11})$$

Here the integrals g, h were defined in (3.2) while the remaining integrals L, E and H can be found in [1]. The expressions for the integrals g, h, L are known in a closed analytic form. The integrals E and H on the other hand are not currently known analytically. However, for our purposes we only need their asymptotic behaviour at small u and the first few terms of their expansion in powers of \bar{v} .

The limit $u \rightarrow 0, \bar{v} \rightarrow 0$ can be described as the limit $x_1 \rightarrow x_2$. For the E - and H -integrals, we apply the well-known formulae [45] for their asymptotic expansion in the limit $x_1 \rightarrow x_2$ typical

¹⁸Furthermore, by considering only the limit $u \rightarrow 0$ with v finite, we are restricted to the sector where only the twist-two operators together with their superdescendants survive. Here it is known that there is a unique twist two, spin S supermultiplet for each even S .

of Euclidean space written in terms of a sum over certain subgraphs of a given graph¹⁹. Making use of conformal invariance, we set $x_4 \rightarrow \infty$, $x_1 = 0$ and arrive at the problem of analyzing the asymptotic behaviour of the Feynman integrals depending on the two external coordinates x_2 and x_3 in the limit $x_2 \rightarrow 0$. The conformal ratios take the form $u = x_2^2/x_3^2$ and $v = x_{23}^2/x_3^2$ so that for $x_2 \rightarrow \rho x_3$ with $\rho \rightarrow 0$ they scale in the Euclidean space as $u = O(\rho^2)$ and $\bar{v} = O(\rho)$. We have evaluated terms up to order $O(\rho^4)$ for all integrals entering the right-hand side of (C.11) and obtained the following results for the E -integrals

$$\begin{aligned}
x_{13}^2 x_{24}^2 E(1, 3; 2, 4) &= \ln u \left[\left(\frac{3}{800} \zeta_3 + \frac{17833}{5529600} \right) \bar{v}^4 + \left(\frac{3}{512} \zeta_3 + \frac{19}{6144} \right) \bar{v}^3 + \left(\frac{1}{96} \zeta_3 + \frac{11}{4608} \right) \bar{v}^2 + \frac{3}{128} \zeta_3 \bar{v} + \frac{3}{32} \zeta_3 \right] \\
&\quad + \left(\frac{1261}{576000} \zeta_3 - \frac{1060073}{82944000} \right) \bar{v}^4 + \left(\frac{5}{9216} \zeta_3 - \frac{1423}{110592} \right) \bar{v}^3 - \left(\frac{5}{1152} \zeta_3 + \frac{37}{3456} \right) \bar{v}^2 - \frac{3}{128} \zeta_3 \bar{v} - \frac{3}{16} \zeta_3, \\
x_{13}^2 x_{24}^2 E(1, 4; 2, 3) &= \ln u \left[\left(\frac{137}{3200} \zeta_3 + \frac{28633}{5529600} \right) \bar{v}^4 + \left(\frac{25}{512} \zeta_3 + \frac{25}{6144} \right) \bar{v}^3 + \left(\frac{11}{192} \zeta_3 + \frac{11}{4608} \right) \bar{v}^2 + \frac{9}{128} \zeta_3 \bar{v} + \frac{3}{32} \zeta_3 \right] \\
&\quad + \left(\frac{659}{144000} \zeta_3 - \frac{1682573}{82944000} \right) \bar{v}^4 - \left(\frac{65}{9216} \zeta_3 + \frac{1865}{110592} \right) \bar{v}^3 - \left(\frac{1}{36} \zeta_3 + \frac{37}{3456} \right) \bar{v}^2 - \frac{9}{128} \zeta_3 \bar{v} - \frac{3}{16} \zeta_3, \\
x_{13}^2 x_{24}^2 E(1, 2; 3, 4) &= \left(\frac{2419}{2073600} \bar{v}^4 + \frac{13}{9216} \bar{v}^3 + \frac{25}{13824} \bar{v}^2 + \frac{1}{384} \bar{v} + \frac{1}{192} \right) (\ln u)^3 \\
&\quad + \left(-\frac{24679}{4608000} \bar{v}^4 - \frac{125}{18432} \bar{v}^3 - \frac{259}{27648} \bar{v}^2 - \frac{1}{64} \bar{v} - \frac{3}{64} \right) (\ln u)^2 \\
&\quad + \left(\frac{10888367}{1244160000} \bar{v}^4 + \frac{209}{18432} \bar{v}^3 + \frac{1361}{82944} \bar{v}^2 + \frac{1}{32} \bar{v} + \frac{5}{32} \right) \ln u \\
&\quad + \left(-\frac{1}{16} \zeta_5 + \frac{29383}{1728000} \zeta_3 + \frac{28271621}{7464960000} \right) \bar{v}^4 + \left(-\frac{5}{64} \zeta_5 + \frac{101}{4608} \zeta_3 + \frac{1381}{331776} \right) \bar{v}^3 \\
&\quad + \left(-\frac{5}{48} \zeta_5 + \frac{209}{6912} \zeta_3 + \frac{235}{55296} \right) \bar{v}^2 + \left(-\frac{5}{32} \zeta_5 + \frac{3}{64} \zeta_3 \right) \bar{v} - \frac{5}{16} \zeta_5 + \frac{3}{32} \zeta_3 - \frac{5}{32}, \quad (C.12)
\end{aligned}$$

and for the H -integrals

$$\begin{aligned}
x_{13}^2 x_{24}^2 H(1, 3; 2, 4) &= \left(-\frac{1997}{6912000} \bar{v}^4 - \frac{29}{110592} \bar{v}^3 - \frac{1}{10368} \bar{v}^2 + \frac{1}{1536} \bar{v} + \frac{1}{192} \right) (\ln u)^3 \\
&\quad + \left(\frac{51643}{11520000} \bar{v}^4 + \frac{575}{110592} \bar{v}^3 + \frac{217}{41472} \bar{v}^2 - \frac{1}{16} \right) (\ln u)^2 \\
&\quad + \left(-\frac{25959283}{1244160000} \bar{v}^4 - \frac{36013}{1327104} \bar{v}^3 - \frac{8623}{248832} \bar{v}^2 - \frac{15}{512} \bar{v} + \frac{9}{32} \right) \ln u \\
&\quad + \left(-\frac{1997}{144000} \zeta_3 + \frac{380181271}{12441600000} \right) \bar{v}^4 + \left(-\frac{29}{2304} \zeta_3 + \frac{344443}{7962624} \right) \bar{v}^3 \\
&\quad + \left(-\frac{1}{216} \zeta_3 + \frac{48113}{746496} \right) \bar{v}^2 + \left(\frac{1}{32} \zeta_3 + \frac{45}{512} \right) \bar{v} + \frac{1}{4} \zeta_3 - \frac{7}{16}, \\
x_{13}^2 x_{24}^2 H(1, 4; 2, 3) &= \left(\frac{19253}{6912000} \bar{v}^4 + \frac{119}{36864} \bar{v}^3 + \frac{79}{20736} \bar{v}^2 + \frac{7}{1536} \bar{v} + \frac{1}{192} \right) (\ln u)^3 \\
&\quad + \left(-\frac{1628267}{69120000} \bar{v}^4 - \frac{3155}{110592} \bar{v}^3 - \frac{371}{10368} \bar{v}^2 - \frac{3}{64} \bar{v} - \frac{1}{16} \right) (\ln u)^2 \\
&\quad + \left(\frac{103801967}{1244160000} \bar{v}^4 + \frac{136573}{1327104} \bar{v}^3 + \frac{33173}{248832} \bar{v}^2 + \frac{95}{512} \bar{v} + \frac{9}{32} \right) \ln u \\
&\quad + \left(\frac{19253}{144000} \zeta_3 - \frac{282347651}{3110400000} \right) \bar{v}^4 + \left(\frac{119}{768} \zeta_3 - \frac{928859}{7962624} \right) \bar{v}^3 \\
&\quad + \left(\frac{79}{432} \zeta_3 - \frac{119557}{746496} \right) \bar{v}^2 + \left(\frac{7}{32} \zeta_3 - \frac{125}{512} \right) \bar{v} + \frac{1}{4} \zeta_3 - \frac{7}{16}, \\
x_{13}^2 x_{24}^2 H(1, 2; 3, 4) &= \ln u \left[\left(-\frac{13}{960} \zeta_3 + \frac{149}{138240} \right) \bar{v}^4 + \left(-\frac{1}{64} \zeta_3 + \frac{1}{768} \right) \bar{v}^3 + \left(-\frac{1}{64} \zeta_3 + \frac{1}{768} \right) \bar{v}^2 + \frac{3}{16} \zeta_3 \right] \\
&\quad + \left(\frac{217}{11520} \zeta_3 - \frac{791}{276480} \right) \bar{v}^4 + \left(\frac{1}{32} \zeta_3 - \frac{1}{256} \right) \bar{v}^3 + \left(\frac{7}{128} \zeta_3 - \frac{7}{1536} \right) \bar{v}^2 + \frac{3}{32} \zeta_3 \bar{v} - \frac{3}{16} \zeta_3. \quad (C.13)
\end{aligned}$$

¹⁹ To find all the contributions to the asymptotic expansion in an automatic way we prefer to use the code `asy.m` [46] which reveals the contributions in the language of regions. Observe that this code works not only for limits typical of Euclidean space but also for other limits, in particular, for limits of Sudakov type.

Substituting these expressions into (C.10) and going through the steps described above we obtain the following results for the three-loop anomalous dimensions $\gamma_S(a)$ (for $S = 0, 2, 4$)

$$\begin{aligned}\gamma_0(a) &= 3a - 3a^2 + \frac{21}{4}a^3 + O(a^4), \\ \gamma_2(a) &= \frac{25}{6}a - \frac{925}{216}a^2 + \frac{241325}{31104}a^3 + O(a^4), \\ \gamma_4(a) &= \frac{49}{10}a - \frac{45619}{9000}a^2 + \frac{300642097}{32400000}a^3 + O(a^4),\end{aligned}\tag{C.14}$$

and for the corresponding coefficients $A_S(a)$

$$\begin{aligned}A_0(a) &= -a + a^2 \left(\frac{3\zeta_3}{2} + \frac{7}{2} \right) - a^3 \left(2\zeta_3 + \frac{25\zeta_5}{4} + 12 \right) + O(a^4), \\ A_2(a) &= -a \frac{205}{1764} + a^2 \left(\frac{5\zeta_3}{28} + \frac{76393}{148176} \right) - a^3 \left(\frac{1315\zeta_3}{5292} + \frac{125\zeta_5}{168} + \frac{242613655}{112021056} \right) + O(a^4), \\ A_4(a) &= -a \frac{553}{54450} + a^2 \left(\frac{7\zeta_3}{440} + \frac{880821373}{17249760000} \right) \\ &\quad - a^3 \left(\frac{520093\zeta_3}{26136000} + \frac{35\zeta_5}{528} + \frac{1364275757197}{5692420800000} \right) + O(a^4).\end{aligned}\tag{C.15}$$

We verified that the relations (C.14) are in agreement with the results of [24]. Notice that $\gamma_0(a)$ coincides with the anomalous dimension of the Konishi operator $\gamma_K(a)$, Eq. (1.1), and the expansion coefficients of $A_0(a) = \sum_{\ell \geq 1} \alpha^{(\ell)} a^\ell$ coincide with the coefficients $\alpha^{(\ell)}$ in (2.16).

Finally, we should note that it is possible to invert this entire process, namely to use predictions for the normalization and anomalous dimensions of operators, and plug them into the conformal partial wave expansion (C.4) to obtain predictions for the four-point correlation function. More specifically, by using the predicted three-loop, all-spin, twist-two anomalous dimensions of [24], together with two-loop and one-loop twist-two data, and plugging into (C.8), one can obtain the three-loop correlation function in the limit $u \rightarrow 0$ with v finite (all except the finite part as $u \rightarrow 0$ which would require three-loop normalization). Summing the resulting expansion, one obtains a closed analytic form for the correlation function in this limit as a sum of harmonic polylogarithms with argument \bar{v} [37].

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